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# A wind turbine efficiency limit higher than the Lanchester (Betz) limit

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It is shown that the upper limit on the fraction of power that can be extracted from an airstream approaching a wind turbine is about 78% rather than the historical value of 59%. This higher limit is based on the assumption that the wind turbine cannot accelerate the flow at its exit plane above the freestream velocity. Although the air slows at the turbine inlet, the only effect of the flow's turning and emerging at the turbine exit plane is that pressure drops, as required when any flow is diverted into a swirl. The flow implied by the derivation of the historical limit of 59% cannot satisfy both the angular momentum equation and the continuity equation.

## 1. Background

The efficiency of a wind turbine system normally includes the aerodynamic efficiency of the blades, the efficiency of the gearbox, if any, and that of the generator. The aerodynamic efficiency referred to in this study is the fraction of power reaching the rotating shaft divided by the total rate at which energy approaches the turbine in the form of wind. This is sometimes called the power coefficient.

As van Kuik (2007) pointed out, the well-known wind turbine aerodynamic efficiency limit of 59% was first published by Lanchester (1915) and then by Joukowsky (1920) and Betz (1920, 1927), so it is called the Lanchester-Betz-Joukowsky limit (see also Bergey 1979). However, the work of Betz was the most available to readers, so the name "Betz limit" was used for some years. For more on the history, see Okulov and van Kuik (2012).

After Hütter (1977) suggested that the limit of 59% was not a definite upper limit, and Inglis (1979) pointed out that at least one wind turbine may have achieved an efficiency higher than this limit, Greet (1980) observed that no such limit can be found in a one-dimensional analysis. Rauh and Seelert (1984) agreed that no theoretically grounded optimum for the aerodynamic efficiency of windmills existed.

The 1-D method used to derive the 59% upper limit has come to be known as the General Momentum Theory (GMT) or actuator disk theory. It does not account for swirl.

The axial induction factor $a$—the amount by which the freestream wind at velocity $v_\infty$ is slowed to $v_1$ at the inlet plane of the wind turbine—is defined as

$$a = \frac{(v_\infty - v_1)}{v_\infty}.$$

As will be shown below, the Lanchester-Betz derivation of the 59% efficiency limit yields the value $a = 1/3$. Most questions about the velocity at the actuator disc have been addressed by Van Kuick (2020).

The Blade Element Method (BEM) is a more detailed and physically accurate approach that accounts for swirl and divides the rotor blade into small segments (or elements), each element being treated as an independent airfoil that generates lift and drag forces. These forces are then integrated along the entire length of the blade to calculate the overall aerodynamic forces on the rotor.

Between these two methods lies the method proposed here of extending GMT to include swirl, but not dividing the blade into several sections. Sharpe (2004) and Wood (2007) followed this approach of including swirl, but they did not obtain a value for the axial induction factor $a$ nor an upper efficiency limit. To the author's knowledge, no study to date has proposed a sound alternative to the 59% efficiency limit. This is because any new limit obtained by incorporating swirl is presumed to lead to a lower efficiency limit since some energy is diverted into rotational motion. But this reasoning misses the more important questions, which are: What happens to the axial induction factor, and are the inlet and exit velocities identical. Unlike the present study, none of the above studies introduces a physically distinct inlet and exit velocity at the actuator disc plane.

Even when swirl is considered, the axial induction factor $a$ is usually taken from the original derivation (GMT) which yielded $a = 1/3$. Alternatively, it can be assumed to have a fixed radial distribution based on experimental data where its value is treated as an arbitrary parameter that is fine-tuned to match the observed performance data or to satisfy empirical relations derived from wind tunnel tests or field data. However, experimental testing requires that the blades be first built, which somewhat defeats the purpose of the analytical prediction when budgets are limited. As noted by Bourhis *et al.* (2023), the choice of $a$ is often arbitrary, unless of course more detailed models such as Blade Element Momentum Theory (BEMT) or CFD are used and $a$ is obtained as part of the numerical solution.

The contention of this paper is that once consideration of swirl is incorporated into the analysis, there is no longer any reason to hold to the value of $a = 1/3$—an artefact of the initial derivation that ignored swirl. But if not $a = 1/3$, some new assumption must be made.

It seems that the most optimistic assumption that can be made is that there are no losses through the blade row, and that the velocity magnitude returns to its free-stream value as it reaches the "actuator disc" exit plane (but with a lower pressure). As shown herein, this assumption results in the value $a = 0.18$. Its justification may be novel, but the only justification that can be given for $a = 0.33$ traces its origins back to the original derivation that cannot satisfy the angular momentum equation. Studies measuring induction factors on built wind turbines in wind tunnels report $a \approx 0.13$ (Neff and Meroney 1997), $0.122 \lesssim a \lesssim 0.165$ (Lindenburg 2003), and the range ($0.2 \lesssim a \lesssim 0.34$) over a blade span (Fritz *et al.* 2024).

For discussion of some inefficiencies of wind turbines which are not discussed here, see Georgiou and Theodoropoulos (2011).

## 2. Analysis

Betz's 1928 paper imagined a design wherein drag is minimized to a negligible amount, and the torque of the wind turbine is produced entirely by lift rather than drag. Referring to Figure 1, we see that the power extracted from the wind turbine is

$$\dot{W} = \frac{\dot{m}}{2}({v_\infty}^2 - {v_3}^2)\,, \tag{1}$$

where the mass flow rate $\dot{m}$ is

$$\dot{m} = \rho\, v_N\, A\,, \tag{2}$$

$A$ is any section of the overall wind turbine streamtube, and $v_N$ is the component of air velocity normal to $A$. The velocity just upstream of the wind turbine is $v_1$, and that just downstream of it is $v_2$. Also, $A_1 \approx A_2$ is the circular area swept by the blades.

Apply the momentum equation to the overall control volume in the figure, remembering to include the reaction force $F_R$ at the wind turbine support, and obtain:

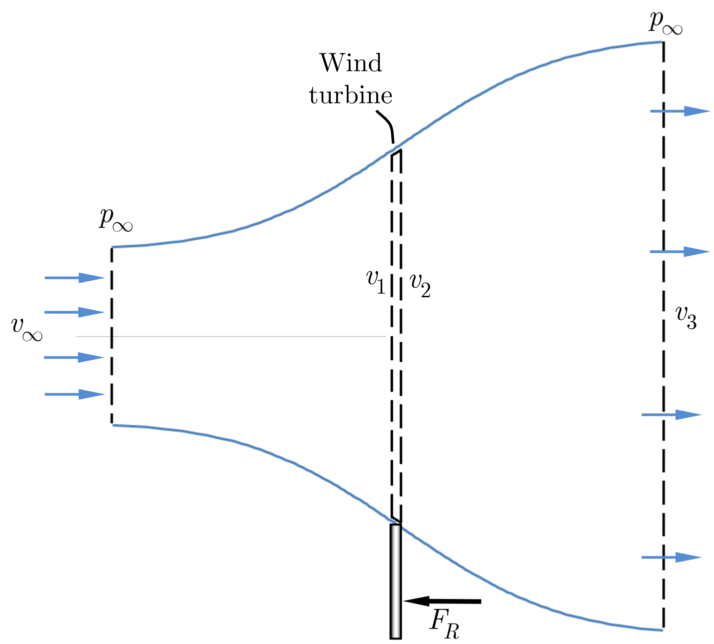



Figure 1. Control volumes used in analysis of wind turbine.

$$F_R = \dot{m}(v_3 - v_\infty)\,. \tag{3}$$

Note that $p_3 = p_\infty$, so these pressure terms dropped out.

Likewise, apply the momentum equation to the small control volume to obtain

$$F_R \;+\; p_1 A_1 \;-\; p_2 A_2 \;=\; \dot{m}(v_{2N} - v_1)\,.$$

For the axial velocity component $v_{2N}$ of $v_2$, we can say that $v_{2N} = v_1$, so the reaction force in the streamwise direction simplifies to

$$F_R \;=\; (p_2 - p_1)A\,. \tag{4}$$

Now eliminate $F_R$ from (3) and (4), and use

$$\dot{m} = \rho\, v_1\, A \tag{5}$$

to obtain

$$\rho v_1 (v_\infty - v_3) \;=\; (p_1 - p_2)\,. \tag{6}$$

This is a useful result that will be used to replace pressures with velocities.

Now apply Bernoulli's equation between the upstream (freestream) plane and plane 1:

$$p_\infty + \frac{1}{2}\rho {v_\infty}^2 \;=\; p_1 + \frac{1}{2}\rho {v_1}^2\,, \tag{7}$$

and also apply Bernoulli's equation between planes 2 and 3 to obtain:

$$p_2 + \frac{1}{2}\rho {v_2}^2 \;=\; p_3 + \frac{1}{2}\rho {v_3}^2\,. \tag{8}$$

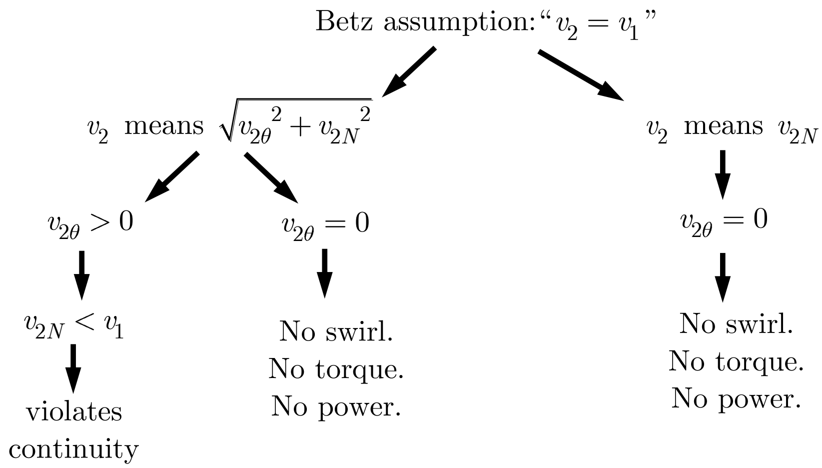
At this point, Betz (and presumably Lanchester and Joukowsky, if not explicitly) made the assumption that

$$v_2 \approx v_1\,. \qquad \text{(Betz)} \tag{9}$$

This allowed him to use (7) and (8) (eliminating $p_3$ and $p_\infty$) to obtain

$$p_2 - p_1 \;=\; \frac{1}{2}\rho({v_3}^2 - {v_\infty}^2)\,. \qquad \text{(Betz)} \qquad (10)$$

However, due to swirl at plane 2, we could write ${v_2}^2 = {v_{2\theta}}^2 + {v_{2N}}^2$ in (8), where $v_{2\theta}$ is the swirl component of velocity at the exit plane. The assumption tree required by the Betz decision $v_2 = v_1$ is shown below:




So if $v_2 = v_1$, then ${v_{2N}}^2 = {v_1}^2 - {v_{2\theta}}^2$, which means that either

$$v_{2N} < v_1 \qquad \text{or} \qquad v_{2\theta} = 0\,, \qquad (11)$$

which violates conservation of mass (left) or requires that there be no swirl (right) and hence no torque produced. But there must be a swirl velocity component $v_{2\theta}$ at plane 2, and the condition $v_{2N} = v_1$ must also hold, by continuity. The following discussion explains what reasoning we can use to allow for this.

Regardless of the how the wind turbine is designed, it will present some obstruction to the free flow of air, so the velocity at plane 1 will be lower than $v_\infty$. And there is no way for the turbine to accelerate the air above $v_\infty$, absent some sort of funneling structure. And since the blades are unconfined by any sort of cowl, this is unlikely. As air passes through the wind turbine, the effect of the blades is merely to turn the flow. As the air emerges to plane 2, the best it can do is regain its original velocity $v_\infty$. So for an upper limit, the following condition is proposed:

$$v_2 = v_\infty\,. \qquad \text{(better)} \qquad (12)$$

(However, the pressure at plane 2 will be lower than at plane 1 due to the extraction of energy by the blades.) Using this limit condition in (8) allows us to calculate the following pressure difference that is consistent with both continuity and angular momentum, unlike (10):

$$p_1 - p_2 \;=\; \rho {v_\infty}^2 \;-\; \frac{1}{2}\rho {v_3}^2 \;-\; \frac{1}{2}\rho {v_1}^2\,. \qquad \text{(better)} \qquad (13)$$

Now the results will differ from those of the historical limit, but for completeness, the remainder of the derivation leading to the 59% will be recounted briefly.

Betz factored $(v_3 - v_\infty)$ out of the right side of (10) and combined it with (6) to obtain

$$v_1 = \frac{1}{2}(v_\infty + v_3). \qquad \text{(Betz)} \qquad (14)$$

Betz then substituted this value of $v_1$ into (5), and the result into (1), to give the following relation for power:

$$\begin{aligned} \dot{W} &= \frac{1}{2}\rho\frac{(v_\infty + v_3)}{2}A({v_\infty}^2 - {v_3}^2) \\ &= \frac{1}{4}\rho A{v_\infty}^3\left(1+\frac{v_3}{v_\infty}\right)\left[1-\left(\frac{v_3}{v_\infty}\right)^2\right] \qquad \text{(Betz)} \qquad (15) \end{aligned}$$

Taking the derivative of $\dot{W}$ with respect to $v_3/v_\infty$ and setting the result to zero indicates that the power will be greatest when the design is such that

$$\frac{v_3}{v_\infty} = \frac{1}{3}. \qquad \text{(Betz)} \qquad (16)$$

Incidentally, this, together with (14), yields an axial induction factor $a$ of

$$a = \frac{(v_\infty - v_1)}{v_\infty} = \frac{1}{3}.$$

Substituting (16) into (15), we compute that the maximum available power from the wind is

$$\dot{W}_{\max} = \frac{16}{27}\frac{\rho}{2}A{v_\infty}^3. \qquad \text{(Betz)} \qquad (17)$$

Therefore, the so-called *Betz limit* says that, at most,

$$\frac{16}{27} \approx 59.3\% \qquad \text{(Betz)}$$

of the oncoming kinetic energy can be converted to useful energy.

As mentioned earlier, we can improve on this estimate, however. We return now to the more accurate pressure difference in (13) and eliminate the pressure terms there, as before, by combining with (6) to obtain the following alternative to (14):

$$v_1 = v_3 - v_\infty + \sqrt{v_\infty(3v_\infty - 2v_3)}. \qquad (18)$$

Substitute (18) into (5), and the result into (1). This gives the following alternative to (16):

$$\dot{W} = \frac{\rho A}{2}[v_3 - v_\infty + \sqrt{v_\infty(3v_\infty - 2v_3)}]({v_\infty}^2 - {v_3}^2).$$

To find a value for $v_3$ that maximizes power, we could differentiate this power with respect to $v_3$, similarly to the strategy of Betz. But the differentiated expression for $\partial\dot{W}/\partial v_3$ becomes too unwieldy. So instead, we can solve for $v_3$ numerically by maximizing power using a spreadsheet solver, and obtain:

$$v_3 = 0.2182v_\infty, \qquad (19)$$

which gives a maximum power of

$$\dot{W}_{\max} = 0.7803\frac{\rho}{2}A{v_\infty}^3. \qquad \text{(better)} \qquad (20)$$

Therefore, at most 78% (not the 59% predicted by (17)) of the energy in the on-coming air stream can be converted into useful work by the turbine.

Substituting (19) into (18) gives the velocity at the turbine inlet plane $v_1$:

$$v_1 \;=\; 0.8193\, v_\infty \,. \tag{21}$$

This yields an axial induction factor $a$ of

$$a \;=\; \frac{(v_\infty - v_1)}{v_\infty} \;\approx\; 0.18\,. \tag{22}$$

Recall that the Lanchester-Betz analysis yielded the value $a = 1/3$.

Substituting (19) and (21) into (6) gives the following relation for the pressure drop across the wind turbine:

$$p_1 - p_2 \;=\; 0.6405\, \rho\, {v_\infty}^2 \,.$$

The thrust can be calculated by $F_R \;=\; \dot{m}(v_{2N} - v_\infty)$, where $v_{2N} = \sqrt{{v_2}^2 - {v_{2\theta}}^2}$. Thus

$$F_R \;=\; \dot{m}\; \sqrt{{v_2}^2 - {v_{2\theta}}^2} - v_\infty \;\;,$$

and by (12),

$$F_R \;=\; \dot{m} v_\infty \left[ \sqrt{1 - \left( \frac{v_{2\theta}}{v_\infty} \right)^2} - 1 \right]. \tag{23}$$

## 3. Blade Twist

The split between axial and swirl velocity at plane 2 depends on the amount of blade turning. If turning is minimal, then we are not extracting as much torque as we could. If turning is maximal, then there is no axial air velocity component, and hence no through flow. In the latter case, power produced would be zero according to (1). Blade twist determines the swirl component $v_{2\theta}$ of the velocity at plane 2.

We can get another equation for the power, other than (1), by using the product of blade rotation rate $\Omega$ and the torque $\mathrm{d}T$ associated with some annulus of the blade passage. The torque $\mathrm{d}T$ created by the air in this annulus is given by the angular momentum equation

$$\mathrm{d}T \;=\; \mathrm{d}\dot{m}\, v_{2\theta}\, r \,. \tag{24}$$

Then, the power $T\,\Omega$ can be equated to (1), as follows:

$$\mathrm{d}\dot{W} \;=\; \mathrm{d}\dot{m} \frac{({v_\infty}^2 - {v_3}^2)}{2} \;=\; \Omega\, \mathrm{d}T \;=\; \Omega\, \mathrm{d}\dot{m}\, v_{2\theta}\, r \,. \tag{25}$$

Solving for the swirl component $v_{2\theta}$ at plane 2 gives:

$$v_{2\theta} \;=\; \frac{{v_\infty}^2 - {v_3}^2}{2\,U} \,. \tag{26}$$

Since blade speed is in the denominator, this gives a profile for $v_{2\theta}$ that is approximately a free vortex. Substituting (19) into (26) gives the optimal swirl velocity

$$v_{2\theta} \;=\; 0.4762 \frac{{v_\infty}^2}{U} \,. \tag{27}$$

Substituting this optimal swirl velocity and $v_1$ from (21) into (25) gives

$$
\begin{aligned}
\mathrm{d}\dot{W}(r) &= \Omega\,\mathrm{d}\dot{m}\,v_{2\theta}\,r \;=\; \Omega\,v_{2\theta}\,\rho\,v_1\,2\pi r^2\,\mathrm{d}r \\
&= 2\pi\rho\,(0.8193v_\infty)\,(0.4762{v_\infty}^2)\,r\,\mathrm{d}r \\
&= 0.7803\,\pi\rho\,{v_\infty}^3\,r\,(r_T - r_R)/n
\end{aligned}
$$

A plot of $\mathrm{d}\dot{W}(r)$ would show that it increases linearly with radius. If the number of radii evaluated between the root radius $r_R$ and the tip radius $r_T$ is, say, $n = 10$, the total power can be computed by the following sum:

$$
\dot{W} \;=\; \sum_{r=r_R}^{r_T} \mathrm{d}\dot{W}(r)\,. \tag{28}
$$

If the differential power is summed at each of the 10 radial locations along the blade, we get a net power very close to the value in (20). For example, if $r_R = 0.1\,\mathrm{m}$, $r_T = 0.8$, $v_\infty = 13\,\mathrm{m/s}$, and $\rho = 1\,\mathrm{kg/m^3}$, (20) gives a power of 1723 W, and the summation in (28) gives 1700 W.

## 4. Pressure at the Wind Turbine Exit Plane

As air passes through the wind turbine, the primary effect of the blades is to turn the flow rather than to slow it down. Since the rotation speed of the retreating blade is slower near the root, the root must turn the oncoming flow more than the tip does. In this way, blade untwist (from root to tip) is dictated by (26). So the swirling flow at the exit plane (plane 2) is much closer to a free vortex than a forced vortex. The exact amount of turning will depend on the final blade twist chosen, but for reference, recall that the velocity in a free vortex is given by:

$$
v_\theta = \frac{k}{r}\,,
$$

where $r$ is some radius along the blade. Applying the following condition at the root radius,

$$
\text{when } r = r_R \quad\rightarrow\quad v_\theta = \Omega_R\,r_R\,,
$$

allows us to write the swirl velocity component as a function of $r$:

$$
v_\theta(r) \;=\; \frac{\Omega_R\,{r_R}^2}{r}\,. \tag{29}
$$

Here, the subscript "$R$" refers to the root of the blade.

Wood (2007) observed that the logarithmic singularity due to the $1/r$ dependence is the fundamental problem in considering the effects of swirl on turbine performance and that it arises from ignoring the structure of the hub vortex. But if we restrict our attention to the annulus just downstream of the blades, we can use the flow field in (29) as the swirl component in the steady momentum equation, written below using Lagrange's decomposition of the acceleration term for convenience:

$$
\underset{0\ \text{(irrot.)}}{\boldsymbol{\omega}\times\mathbf{v}} \;+\; \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) \;=\; -\;\frac{\nabla p}{\rho} \;+\; \underbrace{\nu\left[\nabla^2\mathbf{v} \;+\; \frac{1}{3}\nabla(\nabla\cdot\mathbf{v})\right]}_{0\ \text{(inviscid assumption)}},
$$

which simplifies to

$$
\nabla p \;=\; -\frac{1}{2}\rho\nabla({v_\theta}^2)\,.
$$

Then

$$\frac{\partial p}{\partial r} = -\frac{1}{2}\rho\frac{\partial}{\partial r}({v_\theta}^2)$$

$$\int_{p_R}^{p} \mathrm{d}p = -\frac{1}{2}\rho\int_{r_R}^{r} \mathrm{d}({v_\theta}^2) = -\frac{1}{2}\rho\, {v_\theta}^2\Big|_{r_R}^{r}$$

Substituting the swirl velocity component (29) into this equation gives

$$p - p_R = -\frac{1}{2}\rho\left(\frac{\Omega_R {r_R}^2}{r}\right)^2\Bigg|_{r_R}^{r} = -\frac{1}{2}\rho\,{\Omega_R}^2 {r_R}^4\left(\frac{1}{r^2}-\frac{1}{{r_R}^2}\right)$$

So the radial pressure profile is

$$p(r) = p_R + \frac{1}{2}\rho\,{\Omega_R}^2 {r_R}^2\left[1-\left(\frac{r_R}{r}\right)^2\right].$$

For a swirl velocity with a rotation rate of, say, $\Omega_R = 900$ rpm at the root, the radial pressure profile takes the shape shown in Figure 2. (Note that this is not the rpm of the wind turbine. In fact, it is in the opposite direction of turbine rotation.) The figure shows that the pressure drops at the exit plane due to blade twist, and mostly at the root, in qualitative agreement with velocity measurements made by Neff and Meroney (1997).

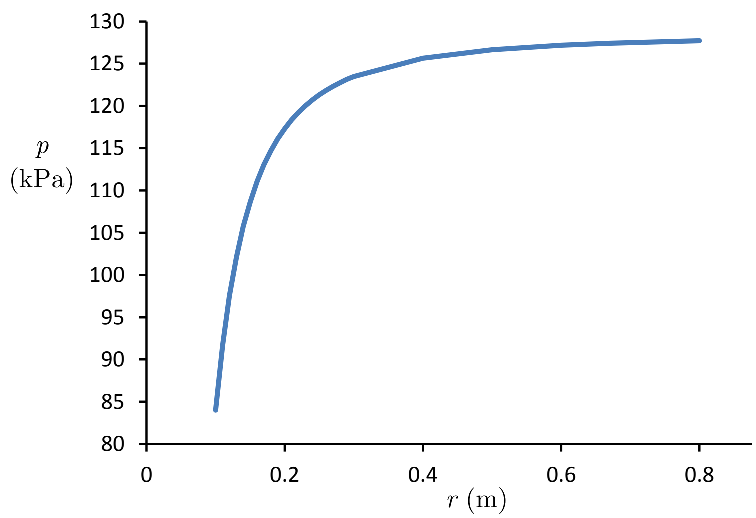



Figure 2. Radial pressure profile for root swirl velocity of $\Omega_R = 94$ rad/s. Other conditions at the root are: $r_R = 0.1\,\mathrm{m}$, $p_R = 84\,\mathrm{kPa}$, and $v_{2\theta} = 9.4\,\mathrm{m/s}$.

## 5. Conclusion

It is remarkable that all three men—Lanchester, Betz, and Joukowsky—while working independently, arrived at the same limit of 59% in a flow that cannot satisfy angular momentum. Although the angular momentum equation (24), which is called Euler's turbine equation, was not needed to derive an upper limit, it must at least be kept in mind in order to avoid the quandary in (11). At the time of Lanchester's work, Euler's turbine equation had already been published in French and German (Bistafa 2021), but it seems to have rarely appeared in English until after 1915 when Lanchester published his limit. This might explain its absence in Lanchester's work. Although Rankine *et al.* (1906, pp. 193-194) and Stodola (1905, p. 28, translation from German) used the concept of Euler's turbine equation and swirl, it was brief and inconspicuous—not highlighted with Euler's name. Other turbine design books written in English before and just after Lanchester's 1915 paper (Blaine 1897; Jude 1906; Garnett 1908; Kennedy 1910; Robinson 1910; Holzwarth 1912; and Church 1928) uniformly omitted any concept of Euler's turbine equation, whether named or unnamed. Incidentally, the most modern thermodynamic treatment of the early works in English was the book by Stodola. By 1930, Euler's turbine equation began appearing in the English literature (see Glauert 1926, p. 210; Paul 1930; and Kearton 1951, pp. 511, 521. It was not in Kearton's 1922 edition, however.) But it was apparently not associated with the name of Euler until Wislicenus (1947, p. 119-121).

The absence of swirl in the Lanchester limit effectively required the implicit assumption of a fictitious extinction of the energy, presumably associated with swirl, a higher dimensional phenomenon. As in (11), it must violate either conservation of axial mass flow or the balance of torque and angular momentum. The 59% limit stems from assuming (12), which simplifies the mathematics but assumes that only pressure changes across the "actuator disc," not velocity. Even after swirl was incorporated, as Betz and other writers later did (Glauert 1926, p. 210; 1935, p. 191), the assumption in (12) persisted.

For the DIY wind turbine designer who cannot run sophisticated numerical simulations, some assumption must be made about the axial induction factor $a$ when doing calculations. The present study effectively determines the axial induction factor $a$ by assuming that as the air passes through the blades, no losses occur. Instead, all kinetic energy from the free stream is recovered fully after the air is turned in the blade passage. Therefore, at the "actuator disk" exit, the magnitude of the air velocity recovers the value it had at the free stream. This assumption yields an axial induction factor of $a \approx 0.18$ and an upper efficiency limit of about 78%.

The swirl far downstream was neglected because the added complexity is not warranted, particularly since other effects, such as dissipation and blade tip losses, are neglected as well.

The work is solely that of the listed author, and author has no conflict of interest.

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