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3-D Dimensional Tolerancing

Thad S. Morton

The formula for the variance of a gap in a three-dimensional stack path is given. A formula is given for the mean of a three-dimensional point cloud centered at the origin when the three rectangular coordinates of each point are normally distributed. A formula is given that relates the variance of a gap in three-dimensional space to the variances of its three rectangular components when those three variances are unequal. A method is described for conducting nonlinear error propagation analysis.

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1. Introduction

The vector loop used in tolerance stackup analysis is well known, but when stack paths are two-dimensional or three-dimensional, engineers are often unfamiliar with the fundamental principle that must be applied. This problem is often mitigated by spot checking a few key dimensions in the assembly model. This paper describes some principles that can be applied to analyze tolerances when stack paths are not confined to one or two dimensions. For a good review of studies related to tolerancing in general, see Hong and Chang (2002).

2. Tolerance Stackup Analysis

Figure 1 shows a hypothetical one-dimensional stack path, or closed loop, for a tolerance stackup analysis. Since all real constructions must allow for variations, a gap must occur somewhere along a closed path formed by mating hardware in an assembly. This closed path is called the *stack path*. Stack paths forming a 3-D vector loop have been depicted graphically by Gao *et al.* (1998) and Yang and Gong (2011), but the formula for the total variance has not been given explicitly. This will be done in (17) below.

If the length of the ith component in an assembly varies in the small tolerance range $\triangle x_i$, then the allowable tolerance of the last component to be inserted into the stack path is dictated by that of all the others. To avoid the possibility of part-to-part variation causing a "no-build" situation on an assembly line, we determine the tolerances of all parts by assuming that the worst possible dimensional combinations allowed on the engineering drawing arrive together on the assembly floor for a specific build. This method of worst-case tolerancing assumes that the gap in an assembly of N components is the following vector sum

$$\sum_{i=1}^{N} \triangle \mathbf{x}_i \;=\; \boldsymbol{\delta} \tag{1}$$

of dimensions $\triangle x_i$ around a closed loop in space along adjoining parts in the assembly. A tolerance stackup analysis consists of two tolerance stacks—one where the $\triangle \mathbf{x}_i$ are part dimensions at tolerance limits corresponding to *maximum material conditions*, and another where the $\triangle \mathbf{x}_i$ are part dimensions at tolerance limits corresponding to *least material conditions*. The way to think of the above summation (the same line integral used in surveying) is with its continuous analogue—the closed line integral:

$$\oint \mathrm{d}\mathbf{x} \;=\; 0 \qquad \text{exact construction (impossible)}$$

$$\oint \mathrm{d}\mathbf{x} \;=\; \boldsymbol{\delta} \qquad \text{real construction} \tag{2}$$

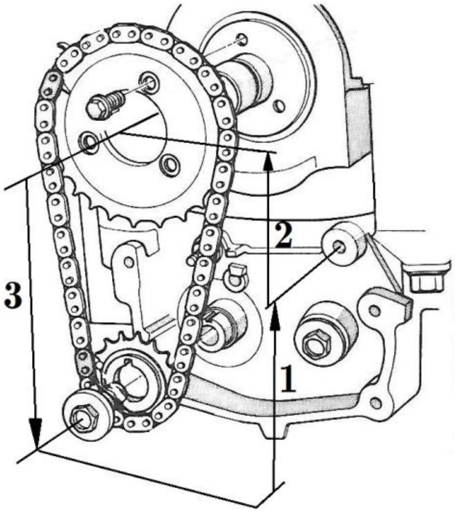

Figure 1. Simplified 1-D stack path in an engine involving the separation distance between sprockets. (Does not include runout of the sprocket or shaft.)

In a perfect world, tolerances would usually all be zero so that any closed loop traversed vectorially through space along part dimensions called out on the engineering drawing would always yield exactly the zero vector if we returned to where we started. But unless all parts in the stack path are press-fit together (an interference fit), the gap δ of a stack path will never be exactly zero.

Consider a gap $\boldsymbol{Y}$ in a stack path containing m part dimensions from some assembly. The width of the gap is given by the vector sum of the dimensions in the stack path:

$$\mathbf{x}_1 + \mathbf{x}_2 + \dots + \mathbf{x}_m + \mathbf{Y} = 0\,. \tag{3}$$

This is referred to as a straight stack. When there are more than $m \approx 6$ or so dimensions in a stack path, it is customary to compute the gap statistically with an L^2 norm instead of using the deterministic vector sum. The variance of some output variable V_Y (which here is a gap in a subassembly of components) that is dependent on m independent input variables (such as all the dimensions in the stack path) is given by:

$$V_Y = V_1 + V_2 + \dots + V_m\,, \tag{4}$$

where V_i is the variance of the ith variable. This statement will be generalized to interdependent input variables in the next section (see (17)), but (4) suffices for many cases. If these variables are normally distributed, then we can express the relation in terms of standard deviations, as follows:

$$s_Y{}^2 = s_1{}^2 + s_2{}^2 + \dots + s_m{}^2 \tag{5}$$

We can use the prediction interval to set the upper and lower tolerance limits of the ith dimension in the stack path:

$$\mathrm{Tol}_i = \pm\, t_i\, s_i \sqrt{\frac{1}{N_i} + 1}\,, \tag{6}$$

where t is the number of standard deviations allowed in the tolerance band in order to achieve the desired probability of success P. (It is the student t-distribution, which is just the dimensionless z-value for small sample size). It is computed with a t-table or with the Excel function "=T.INV((1 - P)/Tails, ndf)," where Tails is either 1 or 2, depending upon whether the specified tolerance is one-sided or two-sided, and ndf is the number of degrees of freedom, which is $N-1$. The complement of the probability P is used in order to get the body of the distribution rather than the tails of the distribution. The result Tol_i is the difference $(x_i - \overline{x}_i)$ between the upper tolerance limit specified on the drawing and its nominal value $\overline{x}_i$. Notice from (6) that in an ideal world, the tolerance limits should depend upon the sample size N used to compute the standard deviation.

Knowing the standard deviation of the gap, the gap tolerance can be related to the fallout rate, or defect rate, allowed for the gap by means of a stackup t-score, t_Y:

$$\mathrm{Tol}_Y = \pm\, t_Y\, s_Y\,. \tag{7}$$

So t_Y is half the number of standard deviations contained in the tolerance band for the gap that will achieve the desired probability of success. By substituting (5) into (7), we have

$$\mathrm{Tol}_Y = \pm\, t_Y \sqrt{s_1^{\,2} + s_2^{\,2} + \ldots + s_m^{\,2}}\,. \tag{8}$$

If all the components in the stack path are designed to the same failure rate as the assembly itself, that is, if each t_i in (6) is equal to t_Y in (8), then (8) can be written as

$$\mathrm{Tol}_Y = \pm \sqrt{(t_1 s_1)^2 + (t_2 s_2)^2 + \ldots + (t_m s_m)^2}\,. \tag{9}$$

If we assume further that all the component tolerances in (6) are based on sample sizes with large N_i, then (6) becomes $\mathrm{Tol}_i \approx \pm\, t_i\, s_i$ so that (9) can be written:

$$\mathrm{Tol}_Y \approx \pm \sqrt{\sum_{i=1}^{m} \mathrm{Tol}_i^{\,2}}\,. \tag{10}$$

In other words, if the t value for the assembly gap is the same as the t values for all the components, then the t values drop out.

To compare the statistical tolerance stackup to the straight stack, we assume for simplicity that all the component tolerances are identical. Then (10) becomes

$$\mathrm{Tol}_Y = \sqrt{m}\, \mathrm{Tol}_i\,, \qquad \text{(statistical stack)} \tag{11}$$

whereas for the straight stack, we have simply

$$\mathrm{Tol}_Y = m\, \mathrm{Tol}_i\,. \qquad \text{(straight stack)} \tag{12}$$

2.1. Bender Safety Factor

To account for the fact processes drift over time and other uncertainties, Bender (1968) proposed a "$6 \times 2.5 = 9$" rule of thumb in tolerance stacking. Bender observed that if there are six independent components, each with a tolerance of ± 2.5, the combined assembly tolerance is not the arithmetic sum of $6(\pm 2.5) = \pm 15$ according to (12), but instead, it is closer to ± 9 (see Figure 2). (Bender pointed out that his lengths 3.5 ± 2.5 were chosen to correspond with the number of sides on the commercially available dice he used in his study.) Meanwhile, the statistical stack according to (11) gives a total tolerance of $\mathrm{Tol}_Y = \sqrt{6}(\pm 2.5) \approx \pm 6.12$. This is why Bender introduced the factor $k = 1.5$ on the right side of (10). Bender stated that the factor 1.5 is necessary because without it, the distributions must all remain perfectly centered. And this is true; however, the reason the 1.5 factor was needed in Bender's study is because he used uniform probability distributions for the 6 components in his assemblies (see Figure 3). If, instead, a normal probability distribution had been used, a factor of approximately $k = 1.1$ would have been appropriate rather than $k = 1.5$. This is illustrated in Figure 4. Still, component lengths can have non-normal distributions, so the assumption of uniform distribution is conservative and seems warranted.

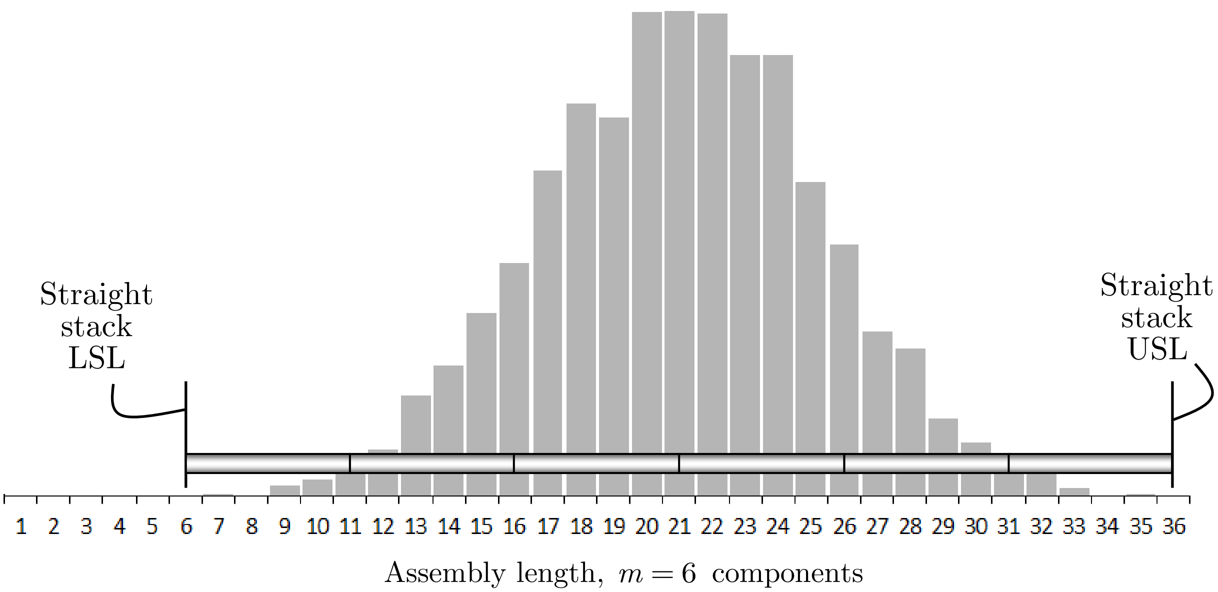

Figure 2. Histogram of the length of a 6-part assembly, with each part of length 3.5 ± 2.5. For comparison, the tolerances of the 6-parts in the assembly are superimposed.

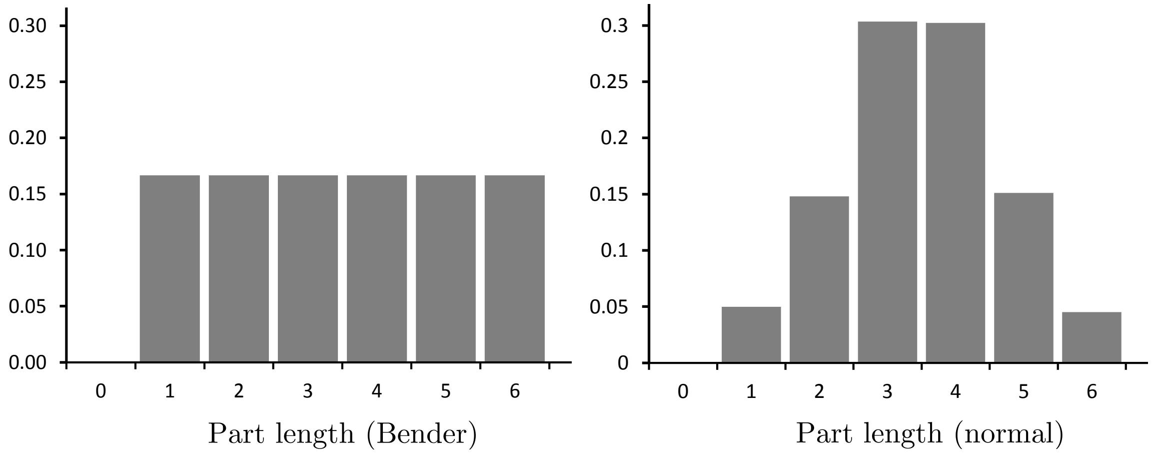

Figure 3. Histogram of Bender's uniformly distributed part (Left) and a normally distributed part (Right) with $\sigma = 1.2$. In both cases (used in Figure 4), $\mu = 3.5$.

Between the curves for the straight stack and the statistical stack shown in Figure 4 are plots from two simulations. Each point in these curves represents a simulation of 3000 points. The reason the defect rate is set to 4.5% (2.25% in each tail) in these simulations is because that is the probability to which Bender's factor $k = 1.5$ effectively corresponds. For comparison purposes, therefore, this probability was preserved for the simulation with normally distributed components.

The 1.5 factor has become the standard in many industries, including the automotive industry. Note that Figure 4 shows that a straight stack involving 4 components yields essentially the same assembly tolerances as the statistical stack of a 7-part assembly.

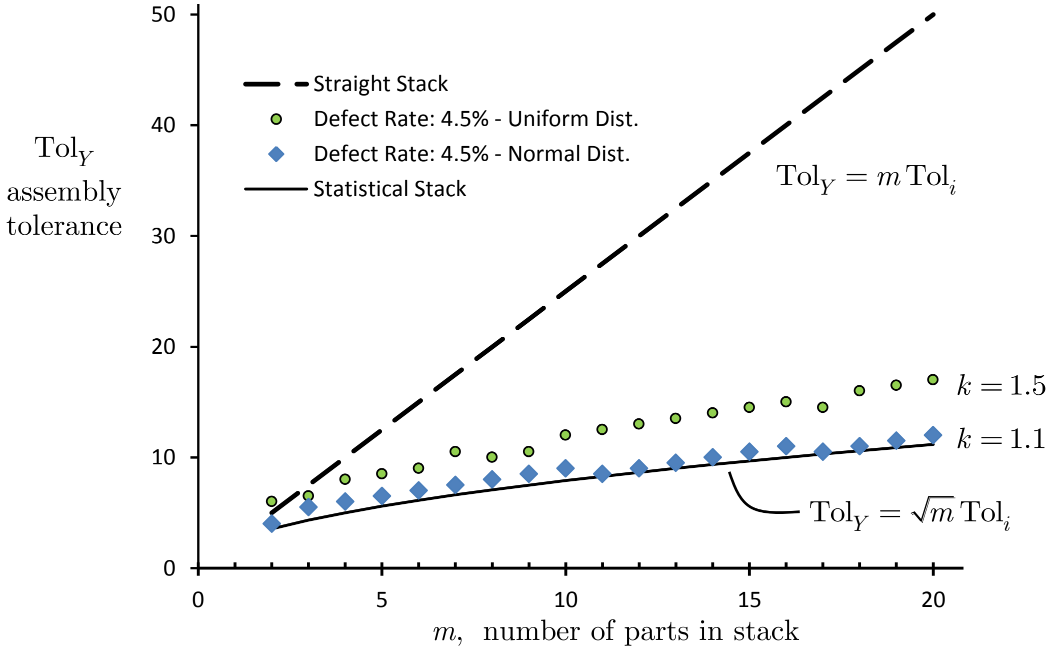

Figure 4. The effect of the $k = 1.5$ factor when component tolerances are all $\mathrm{Tol}_i = \pm 2.5$.

In addition, numerical simulation with components having continuous normal distributions confirms that the defect rate of the assembly will always match that of the components, without the need for the factor k, regardless of how few or how many parts are in the stack. But as noted, since non-normal distributions can arise in components, the more conservative assumption of uniform distribution is probably prudent. But regardless, the crossover point for the two methods is clearly closer to $m = 3$, not $m = 6$. Incidentally, before Bender, Gilson (1951, pp. 45-46) proposed a proportionality constant of $k = 1.6$.

2.2. Statistical Tolerance Stackup

To derive the relation for the gap in a statistical stack, write the differential of (3) as a gap function $\mathbf{Y}(\mathbf{x}_i)$ that spans three-dimensional space ($\mathbf{Y} \in \mathbb{R}^n$ with $n = 3$):

$$\mathrm{d}\mathbf{Y} \;=\; -\frac{\partial \mathbf{Y}}{\partial x_1}\mathrm{d}x_1 \;-\; \frac{\partial \mathbf{Y}}{\partial x_2}\mathrm{d}x_2 \;-\; \dots \;-\; \frac{\partial \mathbf{Y}}{\partial x_m}\mathrm{d}x_m \,. \tag{13}$$

The square of distance transforms as the square of the vector differential in (13) using Riemann's generalization of the Pythagorean Theorem:

$$\mathrm{d}Y^2 \;=\; \mathrm{d}\mathbf{Y}\cdot\mathrm{d}\mathbf{Y} \;=\; \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\frac{\partial Y_j}{\partial x_i}\frac{\partial Y_j}{\partial x_k}\mathrm{d}x_i\,\mathrm{d}x_k \,. \tag{14}$$

Variance also transforms as in (14), which is a generalization of a form proposed by Kline and McClintock (1953) for error analysis. The sample variance of variable x_1 is defined by

$$V_{11} = \sum_{p=1}^{N}\frac{[(x_1)_p - \bar{x}_1]^2}{N-1} \,. \tag{15}$$

Here, $\bar{x}_1$ is the mean (and possibly the nominal) of component 1, and $(x_1)_p$ is the pth measurement or observation of dimension 1. The number N is the number of observations, i.e., the sample size used to estimate the variance.

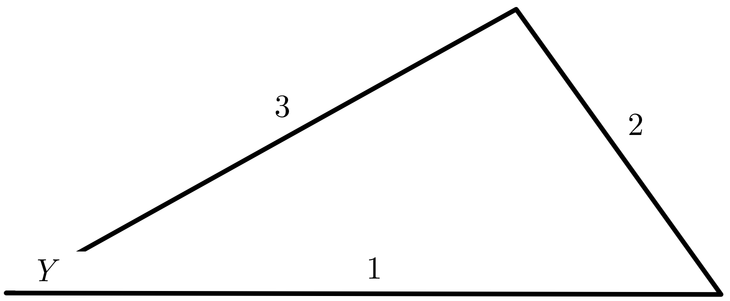

Figure 5. Stack path with 3 mating dimensions in a plane $(n = 2)$.

The generalization of (15) for the rare case when a part dimension can be influenced by the dimension of some other part in the assembly is the *sample covariance* V_{ik}, defined as:

$$V_{ik} = \sum_{p=1}^{N} \frac{[(x_i)_p - \overline{x}_i][(x_k)_p - \overline{x}_k]}{N-1}. \tag{16}$$

Since variance also transforms as in (14), the total variance $V(Y)$ due to the influence of variation of all components x_i on each other is:

$$V(Y) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} \frac{\partial Y_j}{\partial x_i}\frac{\partial Y_j}{\partial x_k} V_{ik}. \tag{17}$$

If we have an assembly of three parts as shown in Figure 5, the indices i and k both range from 1 to $m = 3$. The quantity x_i represents the ith dimension in the stack, and x_k represents the kth dimension in the stack. The random variables x_i in a tolerance stackup will usually be statistically independent of each other so that the off-diagonal components of the covariance matrix will be zero.

If the entire stack path lies in a plane, as does the "assembly" in Figure 5, then $n = 2$, the index j sums only from 1 to 2, and the partial derivatives $\partial Y_1/\partial x_i$ and $\partial Y_2/\partial x_i$ will be the cosine and sine, respectively, of the angle made by dimension i. Since all distance functions in an assembly tolerance stackup are linear, the first derivatives are always constant, and (17) is generally valid. For sensitivity analysis or error propagation with nonlinear functions, see §4.

2.3. Examples

Example 1: 2-D Stackpath, 1 Variable

Suppose that 6 components of equal length $(R = 60\,\text{mm})$ must fit together end-to-end to form a hexagon. The errors in the manufactured lengths of the parts R_1, R_2, R_3,…, R_6 will accumulate as we assemble them end-to-end, as shown in Figure 6. The position of the first end of the first piece (shown as a blue dot) is given a pass because we assume that the location of all the other part interfaces can adjust to the location of its starting point. Also, suppose that the assembly process is such that there is no variation in the angle at which the parts are assembled. The only variation is in the length R of the parts. Figure 6 shows the results of a thousand assemblies of our hexagon, assuming the lengths are normally distributed with standard deviation in length equal to $\sigma_R = 1\,\text{mm}$. Notice that the scatter of the endpoints of $\mathbf{R}_1$ appears in a linear pattern that follows the direction parallel to $\mathbf{R}_1$. Incidentally, the accumulation of scatter in the counterclockwise direction seen in Figure 6 illustrates why it is so important to dimension all of the features from a common datum (coordinate origin) whenever possible.

From the relation

$$R^2 = X^2 + Y^2 \,,$$

we know that the variance of the first endpoint is

$$V_R = {\sigma_R}^2 = {\sigma_X}^2 + {\sigma_Y}^2 \,, \tag{18}$$

where σ_X is the standard deviation of X.

Represent the length of each part in the assembly as $R + \delta R$, the distance R being the mean length. Then the variation can be simulated in Excel with a normal distribution as follows:

$$\delta R = \text{NORMINV(RAND(), 0, } \sigma_R \text{)} \tag{19}$$

Therefore, δR is normally distributed and so can be positive and negative. Then the lengths of the parts are

$$R_1 = R + \delta R_1 \,,$$

$$R_2 = R + \delta R_2 \,,$$

$$:$$

and so on.

The horizontal and vertical components of the end position of the 6th and final component are expressed as summations of all of the others:

$$\left.\begin{aligned} X_6 &= R_1 \cos\theta_1 + R_2 \cos\theta_2 + \ldots + R_6 \cos\theta_6 \\ Y_6 &= R_1 \sin\theta_1 + R_2 \sin\theta_2 + \ldots + R_6 \sin\theta_6 \end{aligned}\right\} \tag{20}$$

As you can see from these relations, errors in the end position of the final component in the assembly will depend on errors of all the other components.

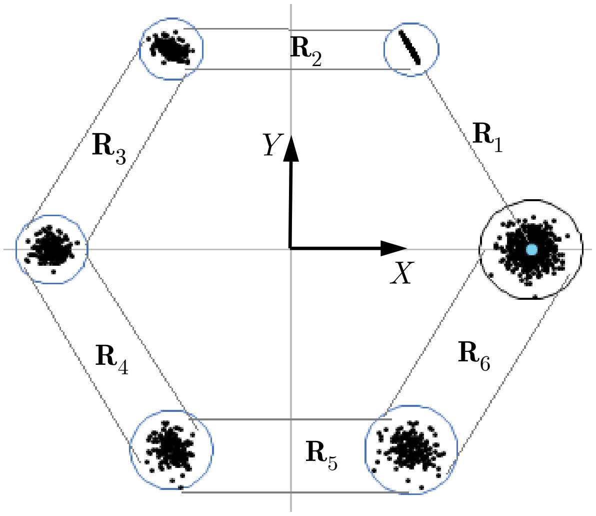

Figure 6. In multi-dimensional problems, variation of a single variable (the length of **R** in this case) can cause variation in multiple output variables (X and Y in this case). $R_i = 60\,\text{mm}$, $\sigma_{R_i} = 1\,\text{mm}$ $(i = 1, 2, \ldots, 6)$. The circles shown correspond to $2.5\sigma_{R_i}$.

Since the variables R and θ are independent (in fact, θ has no variability), the only partial derivatives in (17) that are needed are

$$\frac{\partial X}{\partial R_1} = \cos\theta_1\,, \quad \frac{\partial X}{\partial R_2} = \cos\theta_2\,, \quad \ldots, \ \frac{\partial X}{\partial R_6} = \cos\theta_6$$

$$\frac{\partial Y}{\partial R_1} = \sin\theta_1\,, \quad \frac{\partial Y}{\partial R_2} = \sin\theta_2\,, \quad \ldots, \ \frac{\partial Y}{\partial R_6} = \sin\theta_6$$

To understand the effect that our manufacturing process has on the variance of location of the sixth and final piece in the assembly, we expand (17) and write

$$\begin{aligned} V_{\mathbf{R}} \ = \ & \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial R_1}V_{R_1R_1} + \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial R_2}V_{R_1R_2} + \ldots + \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial R_6}V_{R_1R_6} \\ & + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial R_1}V_{R_1R_2} + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial R_2}V_{R_2R_2} + \ldots + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial R_6}V_{R_2R_6} \\ & \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ & + \frac{\partial X}{\partial R_6}\frac{\partial X}{\partial R_1}V_{R_6R_1} + \frac{\partial X}{\partial R_6}\frac{\partial X}{\partial R_2}V_{R_6R_2} + \ldots + \frac{\partial X}{\partial R_6}\frac{\partial X}{\partial R_6}V_{R_6R_6} \\ & + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial R_1}V_{R_1R_1} + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial R_2}V_{R_1R_2} + \ldots + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial R_6}V_{R_1R_6} \\ & + \frac{\partial Y}{\partial R_2}\frac{\partial Y}{\partial R_1}V_{R_1R_2} + \frac{\partial Y}{\partial R_2}\frac{\partial Y}{\partial R_2}V_{R_2R_2} + \ldots + \frac{\partial Y}{\partial R_2}\frac{\partial Y}{\partial R_6}V_{R_2R_6} \\ & \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ & + \frac{\partial Y}{\partial R_6}\frac{\partial Y}{\partial R_1}V_{R_6R_1} + \frac{\partial Y}{\partial R_6}\frac{\partial Y}{\partial R_2}V_{R_6R_2} + \ldots + \frac{\partial Y}{\partial R_6}\frac{\partial Y}{\partial R_6}V_{R_6R_6} \end{aligned}$$

But since none of the independent variables are correlated with each other, all off-diagonal components of the covariance matrix are zero. Now we input the standard deviations:

$$\begin{aligned} V_{\mathbf{R}} \ = \ & \left(\frac{\partial X}{\partial R_1}\right)^2 {\sigma_{R_1}}^2 + \left(\frac{\partial X}{\partial R_2}\right)^2 {\sigma_{R_2}}^2 + \ldots + \left(\frac{\partial X}{\partial R_6}\right)^2 {\sigma_{R_6}}^2 \\ & + \left(\frac{\partial Y}{\partial R_1}\right)^2 {\sigma_{R_1}}^2 + \left(\frac{\partial Y}{\partial R_2}\right)^2 {\sigma_{R_2}}^2 + \ldots + \left(\frac{\partial Y}{\partial R_6}\right)^2 {\sigma_{R_6}}^2 \end{aligned}$$

Substituting values for the partial derivatives computed above gives

$$\begin{aligned} {\sigma_{\mathbf{R}}}^2 \ = \ & \cos^2\theta_1\,{\sigma_{R_1}}^2 + \cos^2\theta_2\,{\sigma_{R_2}}^2 + \ldots + \cos^2\theta_6\,{\sigma_{R_6}}^2 \\ & + \sin^2\theta_1\,{\sigma_{R_1}}^2 + \sin^2\theta_2\,{\sigma_{R_2}}^2 + \ldots + \sin^2\theta_6\,{\sigma_{R_6}}^2 \end{aligned}$$

Notice that for a linear (in-line or 1D) stack path (where $\cos\theta = \pm 1$ and $\sin\theta = 0$), this simplifies to the conventional one-dimensional tolerance stackup method wherein the cosine functions are replaced by a factor of 1 or -1, depending upon whether they are directed rightward or leftward (or upward or downward). Note that for a straight stack ($m < 7$), these factors do not square to all positive values.

Simplifying further,

$$\sigma_{\mathbf{R}} \ = \ \sqrt{{\sigma_{R_1}}^2 + {\sigma_{R_2}}^2 + {\sigma_{R_3}}^2 + {\sigma_{R_4}}^2 + {\sigma_{R_5}}^2 + {\sigma_{R_6}}^2}\,.$$

Example 2: 2-D Stackpath, 2 Variables

Now suppose that we let both R and θ vary in the manufacture of our product; however, they still vary independently. The length $R+\delta R$ of the part is normally distributed. The distance R is the mean length, and the variation can again be simulated in Excel by:

$$\delta R = \text{NORMINV(RAND(), 0, } \sigma_R \text{)} \tag{21}$$

Therefore, δR is again normally distributed and so can be positive and negative. Then the Cartesian coordinates of the endpoint of the first part, for example, are

$$\left.\begin{aligned} X_1 &= R\cos\theta_1 + \delta R_1 \cos\theta_1 \\ Y_1 &= R\sin\theta_1 + \delta R_1 \sin\theta_1 \end{aligned}\right\} \tag{22}$$

The angles made by the 6 position vectors of the 6 points are $\theta_1 = 0°$, $\theta_2 = 60°$, $\theta_3 = 120°$, $\theta_3 = 180°$, $\theta_4 = 240°$, $\theta_5 = 300°$, and $\theta_6 = 360°$. The angles vary normally such that the arc length $R\delta\theta$ has a standard deviation of 1 mm at the end of each segment. Therefore, $\sigma_\theta = 1\,\text{mm}/R$, and data is simulated with the following Excel formula:

$$\delta\theta = \text{NORMINV(RAND(), 0, } \sigma_\theta \text{)}.$$

The partial derivatives called for in (17) are

$$\begin{array}{llll} \dfrac{\partial X}{\partial R_1} = \cos\theta_1\,, & \dfrac{\partial X}{\partial R_2} = \cos\theta_2\,, & \cdots & \dfrac{\partial X}{\partial R_6} = \cos\theta_6 \\ \dfrac{\partial Y}{\partial R_1} = \sin\theta_1\,, & \dfrac{\partial Y}{\partial R_2} = \sin\theta_2\,, & \cdots & \dfrac{\partial Y}{\partial R_6} = \sin\theta_6 \\ \dfrac{\partial X}{\partial \theta_1} = -R_1\sin\theta_1, & \dfrac{\partial X}{\partial \theta_2} = -R_2\sin\theta_2, & \cdots & \dfrac{\partial X}{\partial \theta_6} = -R_6\sin\theta_6 \\ \dfrac{\partial Y}{\partial \theta_1} = -R_1\cos\theta_1, & \dfrac{\partial Y}{\partial \theta_2} = -R_2\cos\theta_2, & \cdots & \dfrac{\partial Y}{\partial \theta_6} = -R_6\cos\theta_6 \end{array}$$

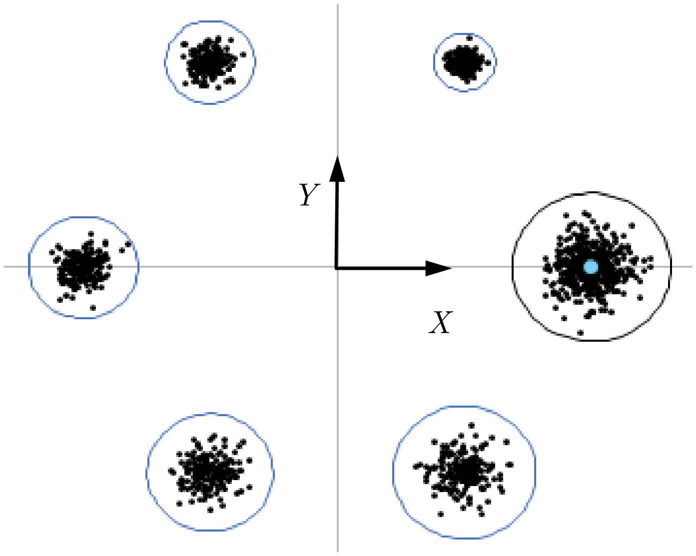

Figure 7. As we start from the Cartesian point $(R,\,0)$ and assemble counterclockwise, the variance of each endpoint position accumulates. Values: $R_i = 60\,\text{mm}$, $\sigma_{R_i} = 1\,\text{mm}$, $\sigma_{\theta_i} = 1/60$. The circles shown correspond to $2.5\sigma_{R_i}$ where $(i = 1,2,...,6)$.

Now we will see the patterns scatter as shown in Figure 7. Expanding (17) gives:

$$\begin{aligned}
V_{\mathbf{R}} \;=\; & \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial R_1}V_{R_1R_1} + \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial R_2}V_{R_1R_2} + \dots + \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial R_6}V_{R_1R_6} \\
& + \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial \theta_1}V_{R_1\theta_1} + \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial \theta_2}V_{R_1\theta_2} + \dots + \frac{\partial X}{\partial R_1}\frac{\partial X}{\partial \theta_6}V_{R_1\theta_6} \\
& + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial R_1}V_{R_2R_1} + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial R_2}V_{R_2R_2} + \dots + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial R_6}V_{R_2R_6} \\
& + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial \theta_1}V_{R_2\theta_1} + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial \theta_2}V_{R_2\theta_2} + \dots + \frac{\partial X}{\partial R_2}\frac{\partial X}{\partial \theta_6}V_{R_2\theta_6} \\
& \qquad\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
& + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial R_1}V_{R_1R_1} + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial R_2}V_{R_1R_2} + \dots + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial R_6}V_{R_1R_6} \\
& + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial \theta_1}V_{R_1\theta_1} + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial \theta_2}V_{R_1\theta_2} + \dots + \frac{\partial Y}{\partial R_1}\frac{\partial Y}{\partial \theta_6}V_{R_1\theta_6} \\
& \qquad\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
& + \frac{\partial Y}{\partial R_6}\frac{\partial Y}{\partial \theta_1}V_{R_6\theta_1} + \frac{\partial Y}{\partial R_6}\frac{\partial Y}{\partial \theta_2}V_{R_6\theta_2} + \dots + \frac{\partial Y}{\partial R_6}\frac{\partial Y}{\partial \theta_6}V_{R_6\theta_6}.
\end{aligned}$$

But again, since none of the independent variables are correlated with each other, all off-diagonal components of the covariance matrix are zero. Therefore,

$$\begin{aligned}
{\sigma_{\mathbf{R}}}^2 \;=\; & \left(\frac{\partial X}{\partial R_1}\right)^2 {\sigma_{R_1}}^2 + \left(\frac{\partial X}{\partial \theta_1}\right)^2 {\sigma_{\theta_1}}^2 + \dots + \left(\frac{\partial X}{\partial R_6}\right)^2 {\sigma_{R_6}}^2 + \left(\frac{\partial X}{\partial \theta_6}\right)^2 {\sigma_{\theta_6}}^2 \\
& + \left(\frac{\partial Y}{\partial R_1}\right)^2 {\sigma_{R_1}}^2 + \left(\frac{\partial Y}{\partial \theta_1}\right)^2 {\sigma_{\theta_1}}^2 + \dots + \left(\frac{\partial Y}{\partial R_6}\right)^2 {\sigma_{R_6}}^2 + \left(\frac{\partial Y}{\partial \theta_6}\right)^2 {\sigma_{\theta_6}}^2,
\end{aligned}$$

which, after the appropriate substitutions, becomes

$$\begin{aligned}
{\sigma_{\mathbf{R}}}^2 \;=\; & \cos^2\theta_1\, {\sigma_{R_1}}^2 + (R_1\sin\theta_1)^2\, {\sigma_{\theta_1}}^2 + \dots + \cos^2\theta_6\, {\sigma_{R_6}}^2 + (R_6\sin\theta_6)^2\, {\sigma_{\theta_6}}^2 \\
& + \sin^2\theta_1\, {\sigma_{R_1}}^2 + (R_1\cos\theta_1)^2\, {\sigma_{\theta_1}}^2 + \dots + \sin^2\theta_6\, {\sigma_{R_6}}^2 + (R_6\cos\theta_6)^2\, {\sigma_{\theta_6}}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sigma_{\mathbf{R}} \;=\; & \sqrt{{\sigma_{R_1}}^2 + {\sigma_{R_2}}^2 + {\sigma_{R_3}}^2 + {\sigma_{R_4}}^2 + {\sigma_{R_5}}^2 + {\sigma_{R_6}}^2 +} \\
& \overline{+ (R_1\,\sigma_{\theta_1})^2 + (R_2\,\sigma_{\theta_2})^2 + (R_3\,\sigma_{\theta_3})^2 + (R_4\,\sigma_{\theta_4})^2 + (R_5\,\sigma_{\theta_5})^2 + (R_6\,\sigma_{\theta_6})^2}
\end{aligned}$$

Notice that the result is the same as if we had taken the rooted sum of squares of the physical components of the variance.

Example 3: 2-D Variation, Correlated Dimensions

An example of a situation wherein two dimensions may be correlated is in the stackup of casting tolerances on a casting drawing and machining tolerances on the subsequent machining drawing (or finish drawing). Often holes that must be machined in a cast housing will be first cast with core pin holes in order to minimize the amount of material that must be removed during machining, thereby reducing cycle time and increasing tool life. Core pin holes also reduce the amount of porosity in the casting wall that is exposed during the machining process. Porosity in a thin-walled housing can allow, for example, detrimental oil seepage out of the machine housing over time.

Suppose we wish to determine a reasonable tolerance for a hole that is to be machined into a casting in which a smaller core pin hole is already cast. The machining equipment will introduce some variation into the process when it brings the drill to the desired hole location. But when the tool begins to actually drill the material, the final hole position will be shifted slightly because the drill will "follow" the position of the cast core-pin hole to some extent.

Suppose that a casting process has demonstrated the capability of casting holes to within a position tolerance of 0.5 mm. And the subsequent machining process has demonstrated the capability to hold hole position to a tolerance of 0.2 mm when drilling a hole without the presence of a core pin hole. However, when drilling into a pre-cast hole, the machinist finds that the tool or drill bit will "follow" the cast hole to some extent. In this case, the final position (X, Y) of the machined hole is influenced by the variability in the cast hole position (x, y). So the two variables are slightly correlated, and the covariance of the two will be non-zero. In order to apply (17), use the following notation for the input variables:

$x_1 =$ cast position x
$x_2 =$ cast position y
$x_3 =$ machined position x (hole drilled into a fresh (flat) surface)
$x_4 =$ machined position y (hole drilled into a fresh (flat) surface)

The output variables are:

$X =$ final hole position x that has been drilled into a cast hole
$Y =$ final hole position y that has been drilled into a cast hole

and

$$R^2 = X^2 + Y^2 .$$

Applying (17) gives

$$\sigma_R{}^2 = \sigma_X{}^2 + \sigma_Y{}^2$$

$$\begin{aligned}
\sigma_R{}^2 = {} & \frac{\partial X}{\partial x_1}\frac{\partial X}{\partial x_1}V_{11} + \frac{\partial X}{\partial x_2}\frac{\partial X}{\partial x_2}V_{22} + \frac{\partial X}{\partial x_3}\frac{\partial X}{\partial x_3}V_{33} + \frac{\partial X}{\partial x_4}\frac{\partial X}{\partial x_4}V_{44} \\
& + 2\frac{\partial X}{\partial x_1}\frac{\partial X}{\partial x_2}V_{12} + 2\frac{\partial X}{\partial x_1}\frac{\partial X}{\partial x_3}V_{13} + 2\frac{\partial X}{\partial x_1}\frac{\partial X}{\partial x_4}V_{14} + 2\frac{\partial X}{\partial x_2}\frac{\partial X}{\partial x_3}V_{23} \\
& + 2\frac{\partial X}{\partial x_2}\frac{\partial X}{\partial x_4}V_{24} + 2\frac{\partial X}{\partial x_3}\frac{\partial X}{\partial x_4}V_{34} \\
& + \frac{\partial Y}{\partial x_1}\frac{\partial Y}{\partial x_1}V_{11} + \frac{\partial Y}{\partial x_2}\frac{\partial Y}{\partial x_2}V_{22} + \frac{\partial Y}{\partial x_3}\frac{\partial Y}{\partial x_3}V_{33} + \frac{\partial Y}{\partial x_4}\frac{\partial Y}{\partial x_4}V_{44} \\
& + 2\frac{\partial Y}{\partial x_1}\frac{\partial Y}{\partial x_2}V_{12} + 2\frac{\partial Y}{\partial x_1}\frac{\partial Y}{\partial x_3}V_{13} + 2\frac{\partial Y}{\partial x_1}\frac{\partial Y}{\partial x_4}V_{14} + 2\frac{\partial Y}{\partial x_2}\frac{\partial Y}{\partial x_3}V_{23} \\
& + 2\frac{\partial Y}{\partial x_2}\frac{\partial Y}{\partial x_4}V_{24} + 2\frac{\partial Y}{\partial x_3}\frac{\partial Y}{\partial x_4}V_{34}
\end{aligned}$$

Since X is orthogonal to x_2 and x_4, and Y is orthogonal to x_1 and x_3, many of the partial derivatives above are zero. Those remaining have magnitude 1. We are left with

$$\sigma_R = \sqrt{V_{11} + V_{33} + 2V_{13} + V_{22} + V_{44} + 2V_{24}} \tag{23}$$

Since

$$V_{11} + V_{22} = ({\sigma_x}^2 + {\sigma_y}^2)_{\text{cast}} = (\sigma_{R_{\text{cast}}})^2$$

$$V_{33} + V_{44} = ({\sigma_x}^2 + {\sigma_y}^2)_{\text{machined}} = (\sigma_{R_{\text{machined}}})^2$$

we can write (23) as

$$\sigma_R = \sqrt{(\sigma_{R_{\text{cast}}})^2 + (\sigma_{R_{\text{machined}}})^2 + 2(V_{13} + V_{24})} \tag{24}$$

In order to compute the covariance V_{13} in (23) from measured data using (16), we need *paired data*. That is, we need the position of the drill bit just before it contacts the core pin hole region—before it is influenced by the core pin. This data is, of course, not practical to obtain. However, we could measure all other quantities in (23) besides the covariance terms in order to assess them. But to do this, it will be easier if we separate (23) into two separate equations—one for the horizontal position and one for the vertical position, as follows:

$${\sigma_X}^2 = V_{11} + V_{33} + 2V_{13} \tag{25}$$

$${\sigma_Y}^2 = V_{22} + V_{44} + 2V_{24} \tag{26}$$

Now we could measure V_{33} by measuring hole positions machined without a core pin, measure V_{11} by measuring core pin hole positions without machining, and finally, measure hole positions machined into pre-cast holes to obtain σ_X. Then we could solve (25) for V_{13}.

If we were to attempt instead to measure V_{13}, it would be imperative that X-data and Y-data be tracked separately, and not simply the total error in the position R alone. When assessing whether the machining process will leave some raw casting surface exposed (non-cleanup) in the hole threads, it is critical to know whether the machined hole has shifted *toward the center* of the core pin hole or *away from it*. This information will be lost if one only analyzes the resultant position R.

We can simulate the situation with the R programming language. We want two vectors, each with 1000 random numbers. The first vector is to represent the x position of a core pin hole that is cast into a metal housing. The mean of this x position data will be, say, 50.1 mm from some datum, and the variance of the x data is, say, $0.0038\ \text{mm}^2$. A second vector will represent the capability of the machining equipment. It will be the x position obtained by the drill bit just before it begins machining into a surface where a cast hole already exists. Let's say that during this machining trial, the mean of this position was 50 mm and the variance was found to be $0.001\ \text{mm}^2$. Since the machining "follows" the cast core pin hole to some extent, let's assume (again in lieu of test data) that the covariance between the cast hole position and the machining of the hole is $-0.0003\ \text{mm}^2$. For this situation, we can create a hypothetical example with help of the statistical software known as "R" to generate random machining and casting data sets that are correlated to each other with a specified correlation coefficient. To do this, we use the command `mvrnorm`, which means "multi-variate random numbers that are normally distributed." It is a function from an R library called `MASS`, so in order to use it, we must invoke that library first by typing:

```
library(MASS)
```

Then the command to generate a two-dimensional normal distribution with a specified covariance matrix is:

```
x <- mvrnorm(100000, mu=c(50.1,50), matrix(c(0.0038, -0.0003,
-0.0003, 0.001),2))                                          (27)
```

This line generates 100,000 random numbers from a 2 normal distributions where variable 1 has a mean of 50.1, and variable 2 has a mean of 50.0. The covariance matrix is specified next (it must be a positive-definite symmetric matrix). We can check to see how close the results of our simulated random data are to our specified input statistics by using the `var()` command on the newly created data `x`. To compute the variance of `x` with the R programming language, type the function `var(x)`. Here is the type of output it provides:

```
var(x)
              [,1]          [,2]
[1,]  0.0037632600 -0.0002961223
[2,] -0.0002961223  0.0009999631
```

The variance and covariance values we supply as input must satisfy the Schwarz inequality:

$$V_{ik} = \sum_{p=1}^{N} \frac{[(x_i)_p - \overline{x}_i][(x_k)_p - \overline{x}_k]}{N-1}$$

$$\leq \sqrt{\sum_{p=1}^{n} \frac{[(x_i)_p - \overline{x}_i]^2}{N-1}} \sqrt{\sum_{j=1}^{n} \frac{[(x_k)_p - \overline{x}_k]^2}{N-1}} = \sigma_i \, \sigma_k$$

This will make the covariance matrix positive definite.

Then from (23), we can write the variance to be expected if a hole is drilled where a smaller cast hole already exists. For example, the horizontal component of it is

$$\begin{aligned} {\sigma_X}^2 &= V_{11} + V_{22} + 2\,V_{12} \qquad (28) \\ &= 0.0038613 + 0.00093936 + 2(-0.0002822) \\ &= 0.004236 \end{aligned}$$

Therefore, the X-component of the standard deviation of the final machining process will be approximately

$$\sigma_X = 0.065\,\text{mm}.$$

We would follow the same process on the Y position data and obtain σ_Y .

Now, there is one erroneous assumption implicit in (17), and it relates to the commutativity of the processes that create the cross term in (28). It turns out that machining of a hole *never precedes* the casting of a core pin. So the factor of 2 in (28) should have been set to 1. But if the lower diagonal of the covariance matrix in (27) is set to zero (or any number other than the upper diagonal), the data simulated by R remains symmetrical, as evidenced by checking with the `var(x)` command. But regardless, the correct variance is:

$$\begin{aligned} {\sigma_X}^2 &= 0.0038613 + 0.00093936 + (-0.0002822) \\ {\sigma_X}^2 &= 0.004518. \qquad (29) \end{aligned}$$

Therefore, the X-component of the standard deviation of the final machining process will be

$$\sigma_X = 0.067\,\text{mm}.$$

To the variance in (29) would be added other sources of variation, such as the variation between cast datum tabs (used for fixturing the part in the machining center) and the machined datum to which the machined hole may be referenced.

2.4. Non-commuting Processes

We saw above that variance transforms as the square of a vector. But most processes in nature are not commutative. The last example was a case in point. When computing the variance of an output variable based on the variances of its input variables, we must keep commutativity in mind. Suppose that we are examining surface roughness measurement data and trying to understand its impact on the friction coefficient of a surface. Let us say that the surface can be honed, or polished, or both. Let the standard deviation of the surface roughness value for the honing process be σ_1 and that for a polished surface be σ_2. The variance V of the friction coefficient Y of the surface can be found from (17), where the covariance V_{ik} of honing and polishing is computed from the data by (16). In this case, the surface finish of the pth honed surface is $x_{1,p}$, and its mean value is $\bar{x}_1$. Likewise, the surface finish of the pth polished surface is $x_{2,p}$, and its mean value is $\bar{x}_2$. Notice that the data in the numerator of (16) is paired data, so care must be taken that they remain so. The partial derivative $\partial Y/\partial x_1$ in (17) can be thought of as a conversion factor that converts the roughness of a honed surface to the coefficient of friction. The partial derivative $\partial Y/\partial x_2$ does the same for the polished surface. Expanding (17), we could write the variance as a geometric product:

$$\begin{aligned} V(Y) = {} & \left(\frac{\partial Y}{\partial x_1}\right)^2 {\sigma_1}^2(\mathbf{e}_1\mathbf{e}_1) + \left(\frac{\partial Y}{\partial x_2}\right)^2 {\sigma_2}^2(\mathbf{e}_2\mathbf{e}_2) \\ & + \frac{\partial Y}{\partial x_1}\sigma_1 \frac{\partial Y}{\partial x_2}\sigma_2(\mathbf{e}_1\mathbf{e}_2) + \frac{\partial Y}{\partial x_2}\sigma_2 \frac{\partial Y}{\partial x_1}\sigma_1(\mathbf{e}_2\mathbf{e}_1) \end{aligned} \tag{30}$$

The first row of this equation is the standard Pythagorean Theorem which independent standard deviations are known to follow. The two terms in the second row can be combined so that (30) resembles the law of cosines:

$$C^2 = A^2 + B^2 - 2AB\cos\theta\,,$$

which is the equivalent of (28). But the variance of surface finish when polishing a honed surface is likely not the same as it is for honing a polished surface. So the covariance matrix may not be symmetric (nor anti-symmetric). In the example above, the factor of 2 in (28) had to be set to 1 since machining of a hole never precedes the casting of a core pin. However, R apparently cannot account for asymmetry in the covariance matrix. So there is nothing in the conventional interpretation of (30) to accommodate non-commutative processes such as the honing and polishing processes, except possibly through the use of bivectors $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_{21} = \mathbf{e}_2\mathbf{e}_1$ in (30). But if we use a non-commutative basis to accommodate such processes, the only statement that can be made generally is that $\mathbf{e}_1\mathbf{e}_2 \neq \mathbf{e}_2\mathbf{e}_1$.

3. Variances of Distances

The expected value, or mean, for the discrete case is:

$$E[q] = \sum_{i=1}^{N} q_i\, f_i\,, \tag{31}$$

where f_i is the probability of each outcome q_i. In the limit of continuous variables and outcomes, the *probability density function* is used to compute the mean:

$$E[q] = \int q(x) f(x)\,\mathrm{d}x\,. \tag{32}$$

Table 1 provides some useful integrals involving the standard normal probability density function (33) and several means calculated using it. They are instances of (32) for finding the mean of a function.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(-z^2/2)}\,\mathrm{d}z = 1 \tag{33}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{(-z^2/2)}\,\mathrm{d}z = 0 \tag{34}$$

$$\int_{-\infty}^{\infty} |z| \frac{e^{(-z^2/2)}}{\sqrt{2\pi}}\,\mathrm{d}z = \sqrt{\frac{2}{\pi}} = 0.79788 \tag{35}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\, e^{(-z^2/2)}\,\mathrm{d}z = 1 \tag{36}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} z^3\, e^{(-z^2/2)}\,\mathrm{d}z = 0 \tag{37}$$

$$\int_{-\infty}^{\infty} |z|^3 \frac{e^{(-z^2/2)}}{2\pi}\,\mathrm{d}z = \frac{2}{\pi} = 0.63662 \tag{38}$$

Table 1. The means of various powers of z and $|z|$, where z is standard normal, i.e., it has the distribution given in (33). ($\mu = 0$, $\sigma = 1$).

A chi-square (χ^2) variable is a sum of squares, where all X_i are normally distributed:

$$z = \chi^2 = \sum_{i=1}^{n} X_i^2 = X_1^2 + X_2^2 + \dots + X_n^2\,. \tag{39}$$

The general formula for the probability density function for a χ^2 variable with n degrees of freedom was shown by Johnson & Leone (1977, Vol. 1, p. 166) to be

$$f(z) = \frac{2^{-n/2}}{\Gamma(n/2)} z^{n/2-1} e^{-z/2} = \frac{2^{-n/2}}{\Gamma(n/2)} \chi^{n-2} e^{-\chi^2/2}\,. \qquad (z \geq 0) \tag{40}$$

Here, X_1 is a set of random numbers that are normally distributed, X_2 is another set, etc.

3.1. Chi Variables (Norms) with Component Variances All Equal

We saw that the chi-square distribution is given by (40), and that this is the distribution of, for example, R^2. But for measurements of R, such as runout, it would be helpful to have the distribution of R itself. This is called simply the *chi density function.* (Another example of a chi variable is the standard deviation itself.) The norm R of n-dimensional data is given by the following formula:

$$R = \sqrt{X_1^2 + X_2^2 + \dots + X_n^2}\,.$$

Hole position and shaft runout are two such examples of this (with $n = 2$ since the data lies in a plane). The n-dimensional χ density function can be obtained by first noting that in one dimension, the probability density function of R is obtained by taking the pdf in (33) as a starting point. The only difference is that for R in one dimension, our variable of integration is $R = \sqrt{X}$, so the domain of integration of (40) is only the non-negative numbers (even though our measurements of X will be both positive and negative numbers). To compensate for halving the domain of integration, we must include a factor of 2 to the right side. Therefore, to write the pdf of R, we modify the pdf in (33) to

$$f(R) \;=\; \frac{2}{\sqrt{2\pi}}\, e^{-R^2/2}\,. \qquad (R > 0) \tag{41}$$

The generalization of (41) for the norm R of n-dimensional data—that is, where R is

$$R = \sqrt{{X_1}^2 + {X_2}^2 + \ldots + {X_n}^2}$$

and the X_i are normally distributed for all i—is given by incorporating a similar factor of 2 in (40):

$$f(R) \;=\; 2\,\frac{2^{-n/2}}{\Gamma(n/2)}\, R^{n-1}\, e^{-R^2/2}\,. \qquad (R > 0) \tag{42}$$

To the author's knowledge, this general formula for the probability density function of the norm R of n-dimensional data centered at the origin when all its rectangular coordinates X_i are normally distributed has not appeared in the literature. However, we see a corollary to it in the position tolerancing of holes in a plane:

$$\delta \;=\; 2\sqrt{(x_{\text{actual}} - x_{\text{spec}})^2 + (y_{\text{actual}} - y_{\text{spec}})^2}\,.$$

Here, the factor of 2 appears because we can never be sure if a hole will be out of spec to the right or to the left. So any allowance we specify on the drawing for the center of the hole will get magnified by a factor of 2. When $n = 2$, i.e., the data is confined to a plane, (42) is called the Rayleigh probability density function.

As noted in (32), the mean of a function can be written

$$\bar{R} = \int R\, f(R)\,\mathrm{d}R\,. \tag{43}$$

Since R is always positive, $\bar{R} > 0$. Using this equation, the mean of R, when R comes from 1, 2, and 3 dimensional data centered at the origin, is computed in Table 2.

$(m = 1)$	$\bar{R} \;=\; \dfrac{2^{1/2}\sigma}{\Gamma(1/2)}\displaystyle\int_0^\infty R\, e^{-R^2/2}\,\mathrm{d}R \;=\; \sqrt{\frac{2}{\pi}}\,\sigma \;=\; 0.7979\,\sigma$	(44)
$(m = 2)$	$\bar{R} \;=\; \dfrac{\sigma}{\Gamma(1)}\displaystyle\int_0^\infty R^2\, e^{-R^2/2}\,\mathrm{d}R \;=\; \sqrt{\frac{\pi}{2}}\,\sigma \;=\; 1.25331\,\sigma$	(45)
$(m = 3)$	$\bar{R} \;=\; \dfrac{\sigma}{\sqrt{2}\,\Gamma(3/2)}\displaystyle\int_0^\infty R^3\, e^{-R^2/2}\,\mathrm{d}R \;=\; 2\sqrt{\frac{2}{\pi}}\,\sigma \;=\; 1.59577\,\sigma$	(46)

Table 2. The mean of the chi variable $R = \sum_{i=1}^{n} \sqrt{{x_i}^2}$ in $n = 1$, 2, and 3 dimensions. $(\bar{x}_i = 0)$

Generalizing Table 2, we can write a single relation for the mean of the n-dimensional distance R from the origin of centered n-dimensional data:

$$\bar{R} \;=\; \sigma\, \frac{2^{1-n/2}}{\Gamma(n/2)} \int_0^\infty R^n\, e^{-R^2/2}\, \mathrm{d}R\,. \tag{47}$$

The 3D case in (46) and the generalization in (47) appear to have never been published. For convenience, the gamma function in the denominator is tabulated below at some key points (e.g., Weisstein 2003):

$$\begin{aligned} &\Gamma(1/2) = \sqrt{\pi}\,, && \Gamma(1) = 1\,, \\ &\Gamma(3/2) = \sqrt{\pi}/2\,, && \Gamma(2) = 1\,, \\ &\Gamma(5/2) = 3\sqrt{\pi}/4\,, && \Gamma(3) = 2\,. \end{aligned}$$

In most older versions of Excel that do not have a dedicated gamma function such as GAMMA(), it can be implemented by taking the exponential of its GAMMALN() function, which older versions had. So, e.g.,

$\Gamma(n/2)$ = EXP(GAMMALN(n / 2)).

In order to calculate the variances using the formula

$$V \;=\; E[R^2] \;-\; \bar{R}^2, \tag{48}$$

we also compute the following means of R^2 using (42):

$$(m=1) \qquad E[R^2] \;=\; \sigma^2 \frac{2^{1/2}}{\Gamma(1/2)} \int_0^\infty R^2\, e^{-R^2/2}\, \mathrm{d}R \;=\; \sigma^2 \tag{49}$$

$$(m=2) \qquad E[R^2] \;=\; \frac{\sigma^2}{\Gamma(1)} \int_0^\infty R^3\, e^{-R^2/2}\, \mathrm{d}R \;=\; 2\sigma^2 \tag{50}$$

$$(m=3) \qquad E[R^2] \;=\; \frac{\sigma^2}{\sqrt{2}\,\Gamma(3/2)} \int_0^\infty R^4\, e^{-R^2/2}\, \mathrm{d}R \;=\; 3\sigma^2 \tag{51}$$

Then the variances of R for 1, 2, and 3 dimensional data are, by (48),

$$(m=1) \qquad \mathrm{Var}(R) \;=\; (E[R^2] - \bar{R}^2) \;=\; \left(1 \;-\; \frac{2}{\pi}\right)\sigma^2 \;=\; 0.3634\,\sigma^2 \tag{52}$$

$$(m=2) \qquad \mathrm{Var}(R) \;=\; (E[R^2] - \bar{R}^2) \;=\; \left(2 - \frac{\pi}{2}\right)\sigma^2 \;=\; 0.4292\,\sigma^2 \tag{53}$$

$$(m=3) \qquad \mathrm{Var}(R) \;=\; (E[R^2] - \bar{R}^2) \;=\; \left(3 - \frac{8}{\pi}\right)\sigma^2 \;=\; 0.4535\,\sigma^2 \tag{54}$$

Here, σ is the standard deviation of any of the three Cartesian component measurements. These relations are only valid for data that is centered about zero. But they can be used as a quick sanity check when working with position error of machined holes, for example. The relations can also be used when simulating runout data from the x and y position error. The relations above assume that the variances σ^2 in all 3 directions are equal. The case when this is not true will be addressed in next section.

3.2. Chi Variables (Norms) with Unequal Variances

Suppose that a random variable R is defined as the following function of three normally distributed random variables X, Y, and Z:

$$R^2 = X^2 + Y^2 + Z^2$$

and that these three are independent of each other and have a mean of zero. To the author's knowledge, no explicit formula has been published relating the variance of R to the variances of the three independent random variables when their variances are unequal; however, from a Monte Carlo simulation of 30,000 normally distributed X, Y, and Z data points (90,000 data points in all), we can find empirically that

$$\begin{aligned} {\sigma_R}^2 \;=\; & A({\sigma_X}^2 + {\sigma_Y}^2 + {\sigma_Z}^2) \;+\; B(\sigma_X\sigma_Y + \sigma_Y\sigma_Z + \sigma_Z\sigma_X) \\ & +\; C(\sigma_X\sigma_Y\sigma_Z)^{2/3} \end{aligned} \tag{55}$$

The leading coefficient A can be determined by letting $\sigma_X = 1$ and $\sigma_Y = \sigma_Z = 0$ so that $m = 1$ (for 1D data), and using (52). This gives

$$A \;=\; 1 - \frac{2}{\pi} \;=\; 0.3634\,.$$

For 2-dimensional data, we have $m = 2$ and set $\sigma_X = \sigma_Y = 1$ in (55). We then find by equating (55) to (53) that

$$B \;=\; \frac{4}{\pi} - \frac{\pi}{2} \;=\; -\,0.2976\,.$$

Finally, for the case when $m = 3$, we set $\sigma_X = \sigma_Y = \sigma_Z = 1$ in (55) and find by equating (55) to (54) that

$$C \;=\; \frac{3\pi}{2} - \frac{14}{\pi} \;=\; 0.25605\,.$$

Substituting all these values into (55) yields the following relation between the variances of normally distributed measurements in 3 mutually orthogonal directions and the variance of their resultant distance from the origin (which distance cannot be negative):

$$\begin{aligned} {\sigma_R}^2 \;=\; & 0.363({\sigma_X}^2 + {\sigma_Y}^2 + {\sigma_Z}^2) \;-\; 0.298(\sigma_X\sigma_Y + \sigma_Y\sigma_Z + \sigma_Z\sigma_X) \\ & +\; 0.256(\sigma_X\sigma_Y\sigma_Z)^{2/3} \end{aligned} \tag{56}$$

This shows the interesting result that due to the middle term, in two dimensions the standard deviation of R is actually *less than* the standard deviation of its components σ_X and σ_Y. In 2D, if $\sigma_X = \sigma_Y$, (55) gives:

$$\sigma_R \;=\; \sqrt{2 - \frac{\pi}{2}}\,\sigma_X\,. \qquad \text{(2D)} \tag{57}$$

It is important to keep in mind that (56) and (57) are only valid if the data is centered about zero. So it is useful for 3-D position tolerancing or in simulating runout measurements.

Example 4: Hole Position

Suppose the position tolerance specified on the drawing of a shaft subassembly is 0.15 mm and that this is supposed to represent three standard deviations of the population. The requirement is to create a Monte Carlo simulation of thousands of normally distributed Cartesian position "measurements" that meet this requirement by using

$$\delta X \;= \text{NORMINV(RAND(), 0, } s_X\text{)} \tag{58}$$

$$\delta Y \;= \text{NORMINV(RAND(), 0, } s_Y\text{)}. \tag{59}$$

The standard deviation implied by the drawing is $\sigma_R = 0.15/3 = 0.05$ mm, but we cannot use this in our simulation because if the X and Y coordinates are normally distributed, position

$$R \;=\; \sqrt{X^2 \;+\; Y^2}\,. \tag{60}$$

follows a χ distribution. First, we simplify (56) for two dimensions in order to find σ_X and σ_Y, which are normally distributed:

$${\sigma_R}^2 \;=\; 0.363({\sigma_X}^2 + {\sigma_Y}^2) \;-\; 0.298\,\sigma_X\,\sigma_Y\,.$$

If we make the assumption that $\sigma_X = \sigma_Y$ (the next example does not make this assumption), then

$$\sigma_X \;=\; \sigma_Y \;=\; \frac{\sigma_R}{\sqrt{0.428}} \tag{61}$$

$$= 0.076 \text{ mm}$$

This standard deviation can be used to generate normally distributed X and Y data to simulate hole position (or runout).

Example 5: Yoke Position / Shaft Runout

Suppose the standard deviations of two features that determine the position of a yoke attached to a shaft are: $\sigma_X = 0.055$ and $\sigma_Y = 0.040$. Determine the expected standard deviation of shaft runout caused by this position error.

According to (56), the variance of shaft runout will be

$$\begin{aligned} {\sigma_R}^2 \;&=\; 0.363({\sigma_X}^2 + {\sigma_Y}^2) \;-\; 0.298\sigma_X\,\sigma_Y \\ &=\; 0.00102 \end{aligned}$$

So the standard deviation of shaft runout is expected to be:

$$\sigma_R \;=\; 0.032\,\text{mm}.$$

This can again be verified with numerical simulation by creating X and Y data using (58) through (60).

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4. Sensitivity Analysis

It is not uncommon that error propagation analysis must be run on nonlinear functions. The nonlinear function might be something as simple as the volume of a cylinder. Kline and McClintock (1953) proposed the following relation as a means of assessing the influence of uncertainty in measured factors x_i on the uncertainty of a function $F(x_i)$ of those factors:

$$\sigma_F \;=\; \sqrt{\left(\frac{\partial F}{\partial x_1}\right)^2 {\sigma_1}^2 \;+\; \left(\frac{\partial F}{\partial x_2}\right)^2 {\sigma_2}^2 \;+\; \ldots \;+\; \left(\frac{\partial F}{\partial x_n}\right)^2 {\sigma_n}^2}\;. \tag{62}$$

This has become the standard in textbooks describing sensitivity analysis or error propagation; however, for large variances, this relation only holds if the function $F(x_i)$ is a linear function of x_i. For nonlinear functions, Greenwood and Chase (1990) discussed using Monte Carlo simulations or the Hasofer-Lind Reliability Index method (Hasofer and Lind 1974). Evans (1975) presented Taylor series terms up to sixth order for the purpose of doing nonlinear error analysis. However, in order to know which higher-order terms should be used, the form of (14) should be used; that is, $\mathrm{d}Y^2 = \mathrm{d}\mathbf{Y}\cdot\mathrm{d}\mathbf{Y}$. Suppose we have the following nonlinear function F of two random variables x_1 and x_2:

$$F \;=\; {x_1}^2\,{x_2}^2 . \tag{63}$$

Since this is a nonlinear function of the random variables, to relate the standard deviation of the independent variables to that of F, we must use higher moments to compute the variance. We could attempt to obtain it by starting with the conventional two-variable Taylor approximation:

$$\mathrm{d}F \;=\; \frac{\partial F}{\partial x_1}\mathrm{d}x_1 \;+\; \frac{\partial F}{\partial x_2}\mathrm{d}x_2 .$$

But since (63) is quadratic, we include up to second-order terms in the Taylor series:

$$\mathrm{d}F \;=\; \frac{\partial F}{\partial x_1}\mathrm{d}x_1 \;+\; \frac{\partial F}{\partial x_2}\mathrm{d}x_2 \;+\; \frac{1}{2}\frac{\partial^2 F}{\partial {x_1}^2}\mathrm{d}x_1 \;+\; \frac{1}{2}\frac{\partial^2 F}{\partial {x_2}^2}\mathrm{d}x_2 .$$

Abbreviating $\partial f/\partial x_1$ in the above relation as $f_{,1}$ and so forth, and computing $\mathrm{d}F^2$ to get the form for ${\sigma_F}^2$ as done to obtain (14), we obtain

$$\sigma_F \;=\; \sqrt{(F_{,1}\,\sigma_1)^2 \;+\; (F_{,1}\,\sigma_1)^2 \;+\; \frac{1}{4}{F_{,11}}^2\,{\sigma_1}^4 \;+\; \frac{1}{4}{F_{,22}}^2\,{\sigma_2}^4 \;+\; F_{,1}\,F_{,11}\,{\sigma_1}^3 \;+\; F_{,2}\,F_{,22}\,{\sigma_2}^3 \;+\; (F_{,12}\,\sigma_1\,\sigma_2)^2 + \ldots} \tag{64}$$

The last term and all other cross terms are zero since the two random variables are independent of each other, making their correlation coefficient ρ zero. The multiplication table of the standard deviations is:

	σ_1	σ_2
σ_1	$\sigma_1\,\sigma_1$	$\rho\,\sigma_1\,\sigma_2$
σ_2	$\rho\,\sigma_2\,\sigma_1$	$\sigma_2\,\sigma_2$

Figure 8 shows a comparison of the actual standard deviation σ_F of F in (63) with the conventional standard deviation estimate (62) and that obtained by the nonlinear relation (64). Clearly, the latter gives much better agreement with the actual standard deviation than the customary linear relation in (62). Each point in Figure 8 represents a standard deviation computed from 30,000 independently generated data points with a normal distribution.

As Figure 8 shows, to maintain accuracy, the standard deviations of the input variables should be kept within the following limit in terms of the coefficient of variation CV:

$$CV = \sqrt{\left(\frac{\sigma_1}{\mu_1}\right)^2 + \left(\frac{\sigma_2}{\mu_2}\right)^2 + \ldots + \left(\frac{\sigma_n}{\mu_n}\right)^2} \;<\; 1 .$$

5. Conclusion

As given in (17), the variance of an n-dimensional gap in an m-part assembly transforms as the square of distance in an nm-dimensional space. When two distinct dimensions are correlated with each other, their measurements must be retained as paired data, and the question of commutativity should be examined. A formula is given for the mean of a three-dimensional point cloud centered at the origin when its Cartesian components are normally distributed [see (47)]. A formula is given in (56) that relates the variance of a gap in three-dimensional space to the variances of the three independent random Cartesian components when those three variances are unequal. A method is given for determining which higher order terms should be used in nonlinear error propagation.

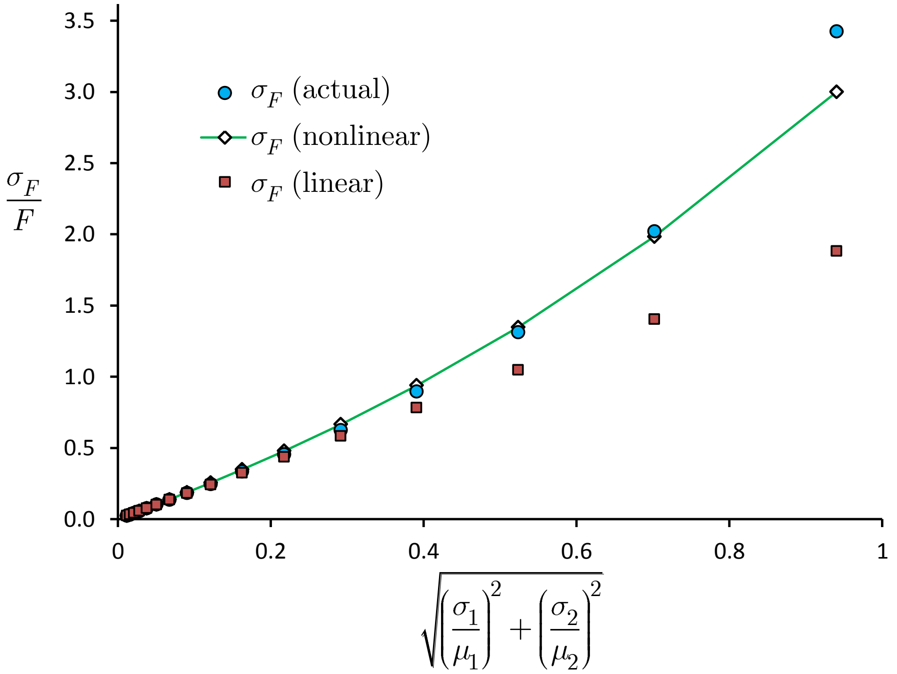

Figure 8. Comparison of the standard deviation of F defined in (63) (with simulated normally distributed data for σ_X and σ_Y) against the linear formula (62) and the nonlinear formula (64) for standard deviation.

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