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Decompositions of Numbers Divisible by 2, 4, and 6

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We decompose quantum versions of numbers of the form $2mM$, $4m$, and $6n$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},$$

be the 2nd quantization of x, so that

$$[2]_q^{\sim} = q + q^{-1},$$
$$[3]_q^{\sim} = q^2 + 1 + q^{-2}.$$

1. A Decomposition Of General Even Numbers, Quantized

Theorem 1. For $m,\ M \in \mathbb{Z}_{\geq 2}$, we have:

$$[2]_q^{\sim}[m]_q^{\sim}[M]_q^{\sim} = [M-m]_q^{\sim} + [M+m]_q^{\sim} + 2\sum_{i=1}^{m-1} \begin{matrix} n \\ k \end{matrix} [M-m+2i]_q^{\sim}. \tag{1}$$

Proof. We are going to use the general formula:

$$[a]_q^{\sim} + [a+2]_q^{\sim} + \ldots + [a+2(m-1)]_q^{\sim} = [m]_q^{\sim}[a+m-1]_q^{\sim},$$

for $a = M - m$ and $a = M - m + 2$.

Thus,

$$[M-m]_q^{\sim} + [M-m+2]_q^{\sim} + \ldots + [M-m+2(m-1)]_q^{\sim} = [m]_q^{\sim}[M-m+m-1]_q^{\sim}$$
$$= [m]_q^{\sim}[M-m+m-1]_q^{\sim} \tag{2}$$
$$[M-m+2]_q^{\sim} + \ldots + [M-m+2(m-1)]_q^{\sim} + [M+m]_q^{\sim} = [m]_q^{\sim}[M+1]_q^{\sim}. \tag{3}$$

The RHS of (1) is the sum of (2) and (3). The useful formula

$$[a]_q^{\sim} + [a+2]_q^{\sim} = [2]_q^{\sim}[a+1]_q^{\sim}.$$

returns

$$[m]_q^{\sim}[M-1]_q^{\sim} + [m]_q^{\sim}[M+1]_q^{\sim} = [m]_q^{\sim}[2]_q^{\sim}[M]_q^{\sim}, \tag{4}$$

which is exactly the left-hand side of (1). ■

2. Decomposition Of Numbers Divisible by Four, Quantized

Theorem 2. For every $m \in \mathbb{Z}$, we have

$$([2]^{\sim}_q)^2 [m]^{\sim}_q \;=\; [m-2]^{\sim}_q \;+\; 2[m]^{\sim}_q \;+\; [m+2]^{\sim}_q \,. \tag{5}$$

Proof. We are going to use the easy formula:

$$[a]^{\sim}_q \;+\; [a+2]^{\sim}_q \;=\; [2]^{\sim}_q [a+1]^{\sim}_q \,. \tag{6}$$

Thus,

$$[m-2]^{\sim}_q \;+\; [m]^{\sim}_q \;=\; [2]^{\sim}_q [m-1]^{\sim}_q \,, \tag{7}$$

$$[m]^{\sim}_q \;+\; [m+2]^{\sim}_q \;=\; [2]^{\sim}_q [m+1]^{\sim}_q \,. \tag{8}$$

Adding up (7) and (8), the right-hand side of (5) becomes:

$$[2]^{\sim}_q \;\; [m-1]^{\sim}_q + [m+1]^{\sim}_q \quad = \; [2]^{\sim}_q [2]^{\sim}_q [m]^{\sim}_q \,,$$

which is the left-hand side of (5). ■

3. Decomposition of Numbers Divisible By 6, Quantized

Theorem 3. For every $m \in \mathbb{Z}$, we have

$$[2]^{\sim}_q [3]^{\sim}_q [m]^{\sim}_q \;=\; [m-3]^{\sim}_q \;+\; 2[m-1]^{\sim}_q \;+\; 2[m+1]^{\sim}_q \;+\; [m+3]^{\sim}_q \,. \tag{9}$$

Proof. We are going to use the two convenient formulae:

$$[a]^{\sim}_q \;+\; [a+2]^{\sim}_q \;=\; [2]^{\sim}_q [a+1]^{\sim}_q \,, \qquad \forall a \,, \tag{10}$$

$$[a]^{\sim}_q \;+\; [a+2]^{\sim}_q \;+\; [a+4]^{\sim}_q \;=\; [3]^{\sim}_q [a+2]^{\sim}_q \,,$$

Thus, we have:

$$[m-3]^{\sim}_q \;+\; [m-1]^{\sim}_q \;+\; [m+1]^{\sim}_q \;=\; [3]^{\sim}_q [m-1]^{\sim}_q \,, \tag{11}$$

$$[m-1]^{\sim}_q \;+\; [m+1]^{\sim}_q \;+\; [m+3]^{\sim}_q \;=\; [3]^{\sim}_q [m+1]^{\sim}_q \,, \tag{12}$$

$$[m-1]^{\sim}_q \;+\; [m+1]^{\sim}_q \;=\; [2]^{\sim}_q [m]^{\sim}_q . \,, \tag{13}$$

The sum of (11) through (13), which is the right-hand side of (9), gives us exactly the left-hand side of (9). ■