J. Sci. Math. Res. (2010) vol. 4, pp. 1-9 Published by Centaurus Press
(Received 2010-07-23; revised 2011-01-09)

Duality for nondifferentiable multiobjective programming involving (Φ, ρ)-univexity

Deo Brat Ojha
Raj Kumar Goel Institute of Technology, Ghaziabad, (U.P.)-INDIA
ojhdb@yahoo.co.in

In this paper we introduced the concepts of (Φ, ρ)-univexity. Here, we considered a dual model associated with a multiobjective programming problem involving support functions. A weak duality result is established under appropriate (Φ, ρ)-univexity conditions.

Key words: (Φ, ρ)-(pseudo/quasi)-convexity/univexity, multiobjective programming, duality theorem.

1. Introduction

For nonlinear programming problems, a number of duals have been suggested, among which the Wolfe dual (Dorn 1960, 1961) is well known. While studying duality under generalized convexity, Mond & Weir (1981) proposed a number of different duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

For $\Phi(x,a,(y,r)) = F(x,a;y) + rd^2(x,a)$, where $F(x,a;.)$ is sublinear on R^n, the definition of (Φ,ρ)-invexity reduces to the definition of (F,ρ)-convexity introduced by Preda (1992), from which Jeyakumar (1985) generalizes the concepts of F-convexity and ρ-invexity. For more details, the reader may consult Chandra & Hussain (1981), Chandra *et al.* (1985, 1998), Chandra & Prasad (1993), Craven (1977), Gulati *et al.* (1997), Unger & Hunter (1974), Preda (1992), Vial (1983), Jeyakumar (1985), Mond & Weir (1981), Hanson & Mond (1982), Weir & Mond (1988), Mishra (1998), Xu (1995), Ojha (2005), Ojha and Mukherjee (2006) for duality under generalized (F,ρ)-convexity, and Liang *et al.* (2003) and Hachimi (2004) for optimality criteria and duality involving (F,α,ρ,d)-convexity or generalized (F,α,ρ,d)-type functions.

The (F,ρ)-convexity was recently generalized to (Φ,ρ)-invexity by Caristi, Ferrara, & Stefanescu (2006), and here we will use this concept to extend some theoretical results of multiobjective programming. Whenever the objective function and all active restriction functions satisfy simultaneously the same generalized invexity at a Kuhn-Tucker point which is an optimum condition, then all these functions should satisfy the usual invexity too. This is not the case in multiobjective programming; Ferrara & Stefanescu (2008) showed that the sufficiency Kuhn-Tucker condition can be proved under (Φ,ρ)-invexity, even if Hanson's invexity is not satisfied (Puglisi 2009).

Therefore, the results of this paper are real extensions of similar results known in the literature. In §2 we define the (Φ, ρ)-univexity. In §3 we consider a class of multiobjective programming problems, and for the dual model we prove a weak duality result and strong duality.

2. Notations and Preliminaries

We denote by R^n the n-dimensional Euclidean space, and by R^n its nonnegative orthant. Further, $R^n_+ = \{x \in R \mid x > 0\}$. For any vectors $x \in R^n$, $y \in R^n$, we denote $x^T y = \sum_{i=1}^n x_i y_i$. Let $C \subset R^n$ be a compact convex set. The support function of C is defined by $s(x \mid C) = \max\{x^T y \mid y \in C\}$. Being convex and everywhere finite, it has a sub differential; that is, there exist $z \in R^n$ such that $s(y \mid C) \geq s(x \mid C) + z^T(y - x)$ for all $y \in C$. The sub differentials of $s(x \mid C)$ is given by $\partial s(x \mid C) = \{z \in C \mid z^T x = s(x \mid C)\}$. For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by $N_D(x) = \{y \in R^n \mid y^T(z - x) \leq 0, \text{ for all } z \in D\}$. For a compact convex set C we obviously have $y \in N_C(x)$ if and only if $s(y \mid C) = x^T y$, or equivalently, if $x \in \partial s(y \mid C)$.

We consider $f : X \to R^p$, $g : X \to R^q$ to be differential functions and $X \subseteq R^n$ to be an open set. We define the following multiobjective programming problem:

(P) minimize

$$f(x) = \left(f_1(x)\text{.........}f_p(x)\right)$$

subject to

$$g(x) \leq 0, x \in X \subseteq R^n$$

Here, '(P) minimize' stands for finding the collection of (properly) efficient points defined below. Let x and y be two vectors in R^n; we use the following convention: $x \leqq y$ if and only if $x_i \leqq y_i, i = 1,2,3,.,.,n; x \leq y$ if and only if $x \leqq y$ but $x \neq y$; $x < y$ if and only if $x_i < y_i;\ i = 1,2,3,.,.,n$. Let X_0 be the set of all feasible solutions of (P); that is, $X_0 = \{x \in X \mid g(x) \leq 0\}$. We quote some definitions and also give some new ones.

Definition 2.1

A vector $a \in X_0$ is said to be an efficient solution of problem (P) if there exits no $x \in X_0$ such that $f(x) \leq f(a), f(x) \neq f(a)$.

Definition 2.2

A point $a \in X_0$ is said to be a weakly efficient solution of problem (P) if there is no $x \in X_0$ such that $f(x) < f(a)$.

Definition 2.3

A point $a \in X_0$ is said to be a properly efficient solution of (P) if it is efficient and there exists a positive constant K such that for each $x \in X_0$ and for each $i \in \{1,2......p\}$ satisfying $f_i(x) < f_i(a)$, there exists at least one $j \in \{1,2......p\}$ such that $f_j(a) < f_j(x)$ and $f_i(a) - f_i(x) \leq K\left(f_j(x) - f_j(a)\right)$.

Denoting by WE(P), E(P), and PE(P) the sets of all weakly efficient, efficient, and properly efficient solutions of (P), we have $\text{PE(P)} \subseteq \text{E(P)} \subseteq \text{WE(P)}$. We denote by $\nabla f(a)$ the gradient vector at a of a differentiable function $f : X \to R$, and by $\nabla^2 f(a)$ the Hessian matrix of f at a. For a real-valued twice differentiable function $\psi(x,y)$ defined on an open set (a,b) in $X \times X$, we denote by $\nabla_x \psi(a,b)$ the gradient vector of ψ with respect to x at (a,b), and by $\nabla_{xx}\psi(a,b)$ the Hessian matrix with respect to x at (a,b). Similarly, we may define $\nabla_{xy}\psi(a,b)$ and $\nabla_{yy}\psi(a,b)$.

Let $f : X \to R$ be a differentiable function $(X \subseteq R^n)$, $X_0 \subseteq X$, and $a \in X$. An element of the $(n+1)$-dimensional Euclidean Space R^{n+1} is represented as the ordered pair (z, r) with $z \in R^n$ and $r \in R$, ρ is a real number, and Φ is a real-valued function defined on $X \times X \times R^{n+1}$, such that $\Phi(x,a,.)$ is convex on R^{n+1} and $\Phi(x,a,(0,r)) \geq 0$ for every $(x,a) \in X \times X$, and $r \in R_+$. Nonnegative functions b_0 and b_1 are defined on $X \times X$. $\psi_0, \psi_1 : R \to R$.

We assume that $\psi_0, \psi_1 : R \to R$ satisfying $u \leq 0 \Rightarrow \psi_0(u) \leq 0$ and $u \leqq 0 \Rightarrow \psi_1(u) \leqq 0$ and $b_0(x,a) > 0$ and $b_1(x,a) \geqq 0$ and $\psi_0(\alpha) = -\psi_0(\alpha)$ and $\psi_1(-\alpha) = -\psi_1(\alpha)$.

Example 2.1

$$\min \ f(x) = x - 1$$

$$g(x) = -x - 1 \leq 0, x \in X_0 \in [1, \infty)$$

$\Phi(x,a;(y,r)) = 2(2^r - 1)|x - a| + \langle y, x - a \rangle$, for $\psi_0(x) = x$, $\psi_1(x) = -x$, $\rho_1 = 1/2$ (for *f*) and $\rho = 1$ (for *g*), then this is (Φ, ρ)-univex, but it is not (Φ, ρ)-invex.

Definition 2.4

A real-valued twice differentiable function $f(.,y) : X \times X \to R$ is said to be (Φ, ρ)-univex at $u \in X$, if for all $b : X \times X \to R_+$, $\Phi : X \times X \times R^{n+1} \to R$, ρ_i is a real number (for i^{th} component of f), we have

$$b(x,u)\left[\psi\{f_i(x,y) - f_i(u,y)\}\right] \geqq \Phi(x,u;(\nabla f_i(u,y), \rho_i)) \tag{2.1}$$

Definition 2.5

A real-valued twice differentiable function $f(.,y) \colon X \times X \to R$ is said to be (Φ, ρ)-pseudounivex at $a \in X$, if for all $b : X \times X \to R_+$, $\Phi \colon X \times X \times R^{n+1} \to R$, ρ_i is a real number (for i^{th} component of f), we have

$$\Phi(x,u;(\nabla f_i(u,y), \rho_i)) \geqq 0 \Rightarrow b(x,u)[\psi\{f_i(x,y) - f_i(u,y)\}] \geqq 0 \tag{2.2}$$

Definition 2.6

A real-valued twice differentiable function $f(.,y) : X \times X \to R$ is said to be (Φ, ρ)-quasi-univex at $a \in X$ with respect to $p \in R^n$, if for all $b : X \times X \to R_+$, $\Phi : X \times X \times R^{n+1} \to R$, ρ_i is a real number (for i^{th} component of f), we have

$$b(x,u)[\psi\{f_i(x,y) - f_i(u,y)\}] \leqq 0 \Rightarrow \Phi(x,u;(\nabla f_i(u,y), \rho_i)) \leqq 0 \tag{2.3}$$

Remark 2.1

(i) If we consider the case $b \equiv 1$, $\Phi(x,u;(\nabla f(u), \rho)) = F(x,u;\nabla f(u))$ (with F sublinear in the third argument, then the above definition reduces to F-convexity).

(ii) When $\Phi(x,u;(\nabla f(u), \rho)) = F(x,u;\nabla f(u)) = \eta(x,u)^T \nabla f(u)$, where $\eta : X \times X \to R^n$, the above definition reduces to η-(pseudo/quasi)-convexity.

A real valued twice differentiable function f is (Φ, ρ)-pseudounicave if $-f$ is (Φ, ρ)-pseudounivex.

3. Mond-Weir Type Symmetric Duality

We consider here the following pair of nondifferentiable multiobjective mathematical programs and establish weak, strong duality theorems.

(MP) Minimize $\{f_i(x,y) + S(x / C_i) - y^T z_i\}$
subject to

$$\sum_{i=1}^{r} \lambda_i [\nabla_y f_i(x,y) - z_i] \leqq 0 \tag{3.1}$$

$$y^T \sum_{i=1}^{r} \lambda_i [\nabla_y f_i(x,y) - z_i] \geqq 0 \tag{3.2}$$

$$z_i \in D_i, i = 1,2,.,.,r \tag{3.3}$$

$$\lambda > 0\,,\ \lambda^T e = 1 \tag{3.4}$$

$$x \geqq 0\,. \tag{3.5}$$

(MD) Maximize $\{f_i(u,v) - S(v / D_i) + u^T w_i\}$
subject to

$$\sum_{i=1}^{r} \lambda_i [\nabla_u f_i(u,v) + w_i)] \geqq 0 \tag{3.6}$$

$$u^T \sum_{i=1}^{r} \lambda_i [\nabla_u f_i(u,v) + w_i)] \leqq 0 \tag{3.7}$$

$$w_i \in C_i, i = 1,2,.,.,r \tag{3.8}$$

$$\lambda > 0\,,\ \lambda^T e = 1 \tag{3.9}$$

$$v \geqq 0\,. \tag{3.10}$$

Here, $e(1,1,1,.,.,1)^T \in R\,, \lambda_i \in R,\ i = 1,2,.,.,r$ and $f_i\,,\ i = 1,2,.,.,r$ are twice differentiable functions from $R^n \times R^n \to R\,,\ D_i$ and $C_i\,,\ i = 1,2,.,.,r$ are compact convex sets in R^n, and $b_i = R^n \times R^m \times R^n \times R^m \to R_+$.

Remark 3.1

Since the objective functions of (MP) and (MD) contain the support functions $s(x|C_i)$ and $s(v|D_i)\,,\ i = 1,2,.,.,p$, these problems are nondifferentiable multiobjective programming problems.

Theorem 3.1 (Weak duality)

Let $(x, y, \lambda, z_1, z_2, .,., z_r)$ be a feasible solution of (MP) and $(u, v, \lambda, w_1, w_2, .,., w_r)$ a feasible solution of (MD) and

(i) $\sum_{i=1}^{r} \lambda_i [f_i(.,v) + (.)^T w_i]$ is (Φ_0, ρ)-univex at u for fixed v,

(ii) $\sum_{i=1}^{r} \lambda_i [f_i(x,.) - (.)^T z_i]$ is (Φ_1, ρ)-unicave at y for fixed x,

(iii) $\Phi_0(x,u;(\xi,\rho)) + u^T\xi \geqq 0$, where $\xi = \sum_{i=1}^{r}\lambda_i[\nabla_u f_i(u,v) + w_i]$, and

(iv) $\Phi_1(v,y;(\zeta,\rho)) + y^T\zeta \leqq 0$, for $\zeta = \sum_{i=1}^{r}\lambda_i[\nabla_y f_i(x,y) - z_i]$,

Then $f_i(x,y) + S(x / C_i) - y^T z_i \not\leq f_i(u,v) - S(v / D_i) + u^T w_i$.

Proof

Since $\sum_{i=1}^{r}\lambda_i[f_i(.,v) + (.)^T w_i]$ is (Φ_0,ρ)-univex at u for fixed v, for $\lambda > 0$, we have

$$\sum_{i=1}^{r}\lambda_i b_o(x,y,u,v)\{\psi[f_i(x,v) + (x)^T w_i] - [f_i(u,v) + (u)^T w_i]\} \geqq \Phi_0(x,u;(\nabla_u f_i(u,v) + w_i,\rho_i)) \quad \text{for all } i = 1,2,.,.,r \tag{3.11}$$

Equation (3.11) provides with the help of hypothesis (iii) and (3.7) with property of b_0 and ψ,

$$\sum_{i=1}^{r}\lambda_i[f_i(x,v) + (x)^T w_i] \geqq \sum_{i=1}^{r}\lambda_i(f_i(u,v) + u^T w_i) \tag{3.12}$$

$x^T w_i \leqq$ S(x / C_i), $i = 1,2,.,.,r$ and from (3.12), then

$$\sum_{i=1}^{r}\lambda_i f_i(x,v) \geqq \left\{\sum_{i=1}^{r}\lambda_i(f_i(u,v) + u^T w_i)\right\} - S(x / C_i) \tag{3.13}$$

Now, $f_i(x,.) - (.)^T z_i$ is (Φ_1,ρ) pseudounicavity assumption at y for fixed x, for $\lambda > 0$, we have,

$$\sum_{i=1}^{r}\lambda_i b_1(x,y,u,v)\{\psi[f_i(x,v) - (v)^T z_i] - [f_i(x,y) + (y)^T z_i]\} \leqq \Phi_1(x,y;(\nabla_y f_i(x,y) - z_i,\rho_i)) \tag{3.14}$$

and hypothesis (iv), (3.2) with property of b_1 and ψ, it implies that

$$\sum_{i=1}^{r}\lambda_i[f_i(x,v) - (v)^T z_i] \leqq \sum_{i=1}^{r}\lambda_i[f_i(x,y) - y^T z_i] \tag{3.15}$$

and $v^T z_i \leqq$ S(v / D_i), $i = 1,2,.,.,r$,

$$\sum_{i=1}^{r}\lambda_i f_i(x,v) \leqq \left\{\sum_{i=1}^{r}\lambda_i(f_i(x,y) - y^T z_i)\right\} + S(v / D_i) \tag{3.16}$$

Combining (3.12) and (3.14), we get

$$\sum_{i=1}^{r}\lambda_i\{f_i(x,y) + S(x / C_i) - y^T z_i\} \geqq \sum_{i=1}^{r}\lambda_i\{f_i(u,v) - S(v / D_i) + u^T w_i\}. \quad \blacksquare$$

Remark 3.2

Following the same lines as in the previous proof, we can easily prove other variants of Theorem 3.1 under the above assumptions, with replacement in the corresponding conditions as below:

(1) the functions $f_i(.,v) + (.)w_i$ are (Φ_0,ρ)-pseudounivex at u.

(2) $f_i(x,.) - (.)z_i$ are (Φ_1,ρ)-pseudounicave at y.

Now, under appropriate conditions, we state a strong duality and a converse duality theorem relative to problems (MP) and (MD).

Theorem 3.2 (Strong duality)

Let $(x', y', \lambda', z_1', z_2', ., ., z_r')$ be a weak efficient solution for (MP) for fixed $\lambda = \lambda'$ in (MD); assume that,

(i) The set $\sum_{i=1}^{r} \lambda_i [\nabla_{yy} f_i]$ is positive or negative definite for all $i = 1, 2, ., ., r$;

(ii) And the set $[\nabla_y f_1 - z_1', \nabla_y f_2 - z_2', ., ., \nabla_y f_r - z']$ for all $i = 1, 2, ., ., r$ is linearly independent.

Then there exist $w' \in R^n{}_i$, $i = 1, 2, ., ., r$, such that $(x', y', \lambda', w_1', w_2', ., ., w_r')$ is a feasible solution of (MD), $b_i(x', y', u', v') > 0, i = 1, 2, ., ., r$, and the two objectives have the same values. Also, if the hypothesis of Theorem 3.1 are satisfied for all feasible solutions of (MP) and (MD), then $(x', y', \lambda', w_1', w_2', ., ., w_r')$ is an efficient solution for (MD).

Proof

Let $(x', y', \lambda', z_1', z_2', ., ., z_r')$ be a weak efficient solution for (MP). Then it is weakly efficient solution. Hence, there exist $\alpha \in R^r$, $\beta \in R^r$, $\gamma \in R^r$, $\mu \in R^r$ and $n \in R$ not all zero and $w_1' \in R^r$, $i = 1, 2, ., ., r$ such that the following Fritz-John optimality condition (28) are satisfied at $(x', y', \lambda', z_1', z_2', ., ., z_r')$.

$$\alpha_i (\nabla_x \{f_i + w_i\}) + (\beta - ny')^T \lambda_i (\nabla_{yy} f_i) = s \tag{3.17}$$

$$w_i \in R^n \text{, for all } i = 1, 2, ., ., r \tag{3.18}$$

$$x'^T w_i' \in S(x' / C_i) \tag{3.19}$$

$$\sum_{i=1}^{r} (\alpha_i - n\lambda_i)[\nabla_y f_i - z_i] + (\beta - ny')^T \lambda' (\nabla_{yy} f_i) = 0 \text{, for all } i = 1, 2, ., ., r \tag{3.20}$$

$$(\beta - ny')^T \lambda' [\nabla_y f_i - z_i] - \mu_i = 0 \text{, for all } i = 1, 2, ., ., r \tag{3.21}$$

$$\alpha_i y' - (\beta - ny')^T \lambda' \in N_{D_i}(z_i') , \tag{3.22}$$

$$\beta^T \sum_{i=1}^{r} \lambda' [\nabla_y f_i - z_i] = 0 \tag{3.23}$$

$$\Rightarrow ny'^T \sum_{i=1}^{r} \lambda' [\nabla_y f_i - z_i] = 0 \tag{3.24}$$

and

$$s^T x' = 0 , \tag{3.25}$$

$$\mu^T \lambda' = 0 \tag{3.26}$$

$$(\alpha, \beta, s, \lambda, \mu, n) \geq 0, \tag{3.27}$$

$$(\alpha, \beta, s, \lambda, \mu, n) \neq 0, \quad \lambda' > 0 \text{ and } \mu \geq 0 \tag{3.28}$$

Equation (3.26) implies $\mu = 0$. Consequently, (3.21) gives

$$(\beta - ny')^T \lambda'[\nabla_{yy} f_i](\beta - ny') = 0 \tag{3.29}$$

hence, in the view of (i),

$$\beta = ny', \tag{3.30}$$

From (3.20) and (3.30) with assumption (ii), we have

$$\alpha_i = n\lambda_i \text{ for all } i = 1,2,.,.,r \tag{3.31}$$

If $n = 0 \Rightarrow \alpha_i = 0, \beta = 0, \mu_i = 0, s = 0$ for all $i = 1,2,.,.,r$, $(\alpha, \beta, s, \lambda, \mu, n) = 0, n > 0$, then we obtain $(\alpha, \beta, s, \lambda, \mu, n) = 0$, which contradicts (3.28), hence $n > 0$.
From (3.31) $\lambda' > 0$ we have $\alpha_i > 0$, $i = 1,2,.,.,r$. From (3.17), (3.30), and (3.31) we get,

$$\alpha_i(\nabla_x \{f_i + w_i\}) = s / n \geq 0. \tag{3.32}$$

By (3.27) and (3.30), since $\eta > 0$, we have

$$y' = \beta / n \geq 0. \tag{3.33}$$

From (3.25) and (3.32), it follows that

$$x'^T \alpha_i(\nabla_x \{f_i + w_i\}) = 0.$$

From (3.18), (3.32), and (3.33), we know that $(x', y', \lambda', w_1', w_2', .,., w_r')$ is feasible for (MD). Now from (3.22) and (3.30), we obtain that

$$y'^T z_i' = S(y' / D_i),\ i = 1,2,.,.,r. \tag{3.34}$$

Using (3.19) and (3.34), we get,

$$f_i(x', y') + S(x' / C_i) - y'^T z_i' = f_i(x', y') - S(y' / D_i) + x'^T z_i',$$

and the objective values of (MD) and (MP) are equal.

We claim that $(x', y', \lambda', w_1', w_2', .,., w_r')$ is an efficient solution for (MD), for if it is not true, then there would exist $(u', v', \lambda', w_1', w_2', .,., w_r')$ feasible for (MD) such that

$$f_i(u, v) - S(v / D_i) + u^T w_i \not\leq f_i(x', y') - S(y' / D_i) + x'^T w_i',\ i = 1,2,.,.,r.$$

Using equalities (3.19) and (3.34), a contradiction (Weak Duality Theorem 3.1) is obtained.

If $(x', y', \lambda', w_1', w_2', .,., w_r')$ is improperly efficient, then for every scalar $M > 0$, there exist a feasible solution $(u', v', \lambda', w_1', w_2', .,., w_r')$ in (MD) and an index i such that

$$\{f_i(u, v) - S(v / D_i) + u^T w_i\} - \{f_i(x', y') - S(y' / D_i) + x'^T w_i'\}$$

$$> M[\{f_j(x', y') - S(y' / D_j) + x'^T w_j'\} - \{f_j(u, v) - S(v / D_j) + u^T w_j\}]$$

for all j satisfying

$$\{f_j(x', y') - S(y' / D_j) + x'^T w_j'\} > \{f_j(u, v) - S(v / D_j) + u^T w_j\}$$

whenever

$$\{f_i(u, v) - S(v / D_i) + u^T w_i\} > \{f_i(x', y') - S(y' / D_i) + x'^T w_i'\}.$$

Since $x'^T w_i' \in S(x' / C_i)$ and $y'^T z_i' = S(y' / D_i)$, $i = 1,2,.,.,r$, it implies that $\{f_i(u, v) - S(v / D_i) + u^T w_i\} > \{f_i(x', y') + S(x / C_i) - y'^T z_i'\}$ can be made arbitrarily large and hence for λ' with $\lambda_i' > 0$, we have

$$\sum_{i=1}^{r}\lambda_i'\{f_i(u,v)-S(v\,/\,D_i)+u^Tw_i\}>\sum_{i=1}^{r}\lambda_i'\{f_i(x',y')+S(x\,/\,C_i)-y'^Tz_i'\},$$

which contradicts the weak duality theorem 3.1. □

4. Special Cases

It has been shown in many earlier works (Chandra *et al.* 1985, Mond *et al.* 1991) that non-smooth programming duality can be tackled by introducing quadratic terms like $(x^TAx)^{1/2}$. Even in the fractional objective case, the numerator as well as denominator can contain such quadratic terms. It is readily shown that $(x^TAx)^{1/2}=S(x\,/\,C)$, where $C=\{Ay:y^TA\le 1\}$, and this set is compact and convex.

(i) If $\Phi(x,u;(\nabla f(u),\rho))=F(x,u;\nabla f(u)$, $C_i=0$ and $D_i=0$, $b=1,\psi\equiv I$, then the (MP) and (MD) reduce to the case dealt with by Weir & Mond (1988).

(ii) If $\Phi(x,u;(\nabla f(u),\rho))=F(x,u;\nabla f(u)$, $(x^TB_ix)^{1/2}=S(x\,/\,C_i)$, where $C_i=\{B_iy:y^TB_i\le 1\}$ and $(x^TC_ix)^{1/2}=S(x\,/\,D_i)$, where $D_i=\{C_iy:y^TC_i\le 1\}$, $b=1,\psi\equiv I$ then the programs reduce to special cases dealt with by Mond *et al.* (1992).

(iii) If $\Phi(x,u;(\nabla f(u),\rho))=F(x,u;\nabla f(u)$ (MP) and (MD), $r=1$, $(x^TB_ix)^{1/2}=S(x\,/\,C_i)$, where $C_i=\{B_iy:y^TB_i\le 1\}$ and $(x^TC_ix)^{1/2}=S(x\,/\,D_i)$, where $D_i=\{C_iy:y^TC_i\le 1\}$, $b=1,\psi\equiv I$, we can get the results of Chandra *et al.* (1985).

(iv) If $\Phi(x,u;(\nabla f(u),\rho))=F(x,u;\nabla f(u)$, for $\rho=0$, $b=1,\psi\equiv I$ (Yang *et al.* 2000).

(v) If $\Phi(x,u;(\nabla f(u),\rho))=F(x,u;\nabla f(u)$, $b=1,\psi\equiv I$ (Ojha and Mukherjee 2006).

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