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The Leibniz formula for the symmetric q-derivatives

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The classical Leibniz formula for multiple derivatives is quantized.

The classical Leibniz formula for multiple derivatives,

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

where

$$g^{(k)} = \sum_{k=0}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (g),$$

is quantized.

Let T stand for the q multiplication of the argument:

$$(T^s f)(x) = f(q^s x), \qquad s \in \mathbb{Z}. \tag{1}$$

Introduce the (symmetric) q-derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}_q^{\sim} x}(x) = \frac{f(Tx) - f(T^{-1}x)}{(q - q^{-1})x}, \tag{2}$$

so that

$$\frac{\mathrm{d}x^s}{\mathrm{d}_q^{\sim} x} = [s]_q^{\sim} x^{s-1}, \qquad s \in \mathbb{Z} \tag{3}$$

where

$$[s]_q^{\sim} = \frac{q^s - q^{-s}}{q - q^{-1}}, \qquad s \in \mathbb{Z} \text{ (or } \mathbb{C}) \tag{4}$$

Denote

$$f^{(n)} = \left(\frac{\mathrm{d}}{\mathrm{d}x_q^{\sim}}\right)^{(n)} (f), \qquad n \in \mathbb{Z}_{\geq 0}. \tag{5}$$

Then the usual Leibniz formula reads

$$(fg)' = f'T(g) + T^{-1}(f)g', \tag{6}$$

with f' standing for $\dfrac{\mathrm{d}f}{\mathrm{d}_q^{\sim}x}$.

Theorem 7. For $n \in \mathbb{Z}_{\geq 0}$, we have:

$$(fg)^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} T^{-k}(f^{(n-k)}) \cdot T^{n-k}(g^{(k)}) \tag{8}$$

Proof. We use induction on n, using the obvious relation

$$(T^r f)' = q^r T^r (f'), \qquad r \in \mathbb{Z}\,. \tag{9}$$

We have:

$$(fg)^{(n+1)} = \sum_{s=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} T^{-s}(f^{(n+1-s)}) \cdot T^{n+1-s}(g^{(s)})$$

$$\overset{?}{=} [(fg)^{(n)}]' = \left\{ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} T^{-k}(f^{(n-k)}) \cdot T^{n-k}(g^{(k)}) \right\}'$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} \left\{ [T^{-k}(f^{(n-k)})]'[T^{n+1-k}(g^{(k)})] + [T^{-k-1}(f^{(n-k)})][T^{n-k}(g^{(k)})]' \right\}$$

$$\begin{aligned} \therefore (fg)^{(n+1)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} \Big\{ & q^{-k}[T^{-k}(f^{(n+1-k)})][T^{n+1-k}(g^{(k)})] \\ & + [T^{-k-1}(f^{(n+1)-(k+1)})][q^{n-k} T^{n+1-(k+1)}(g^{(k+1)})] \Big\} \end{aligned} \tag{10}$$

Thus, (8) amounts to the equality

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} \overset{?}{=} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + q^{n-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim}, \tag{11}$$

which is true.