

On alternating sums of squares of quantum integers

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The classical formula $\sum_{i=0}^n (-1)^{i-1} i^2 = (-1)^{n-1} \sum_{k=0}^n k$ is quantized.

The classical formula in the Abstract can be quantized thusly:

$$\sum_{i=0}^n (-1)^{i-1} \left([i]_q^\sim \right)^2 = (-1)^{n-1} \sum_{i=0}^n [i]_{q^2}^\sim . \quad (1)$$

where

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}} , \quad (2)$$

To prove (1) we set $z_k = q^k - q^{-k}$, notice that (1) is true for $n = 0, 1$, and then proceed by induction. The inductive step $n \rightarrow n + 2$ amounts to:

$$(-1)^n \left([n+1]_q^\sim \right)^2 + (-1)^{n-1} \left([n+2]_q^\sim \right)^2 \stackrel{?}{=} (-1)^{n-1} \left\{ [n+1]_{q^2}^\sim + [n+2]_{q^2}^\sim \right\} ,$$

or

$$-\left([n+1]_q^\sim \right)^2 + \left([n+2]_q^\sim \right)^2 \stackrel{?}{=} [n+1]_{q^2}^\sim + [n+2]_{q^2}^\sim , \quad (3)$$

Call

$$X = q^n .$$

Then (3) becomes

$$\frac{1}{z_1^2} \left\{ -\left(Xq - X^{-1}q^{-1} \right)^2 + \left(Xq^2 - X^{-1}q^{-2} \right)^2 \right\} \stackrel{?}{=} \frac{1}{z_2} \left\{ X^2q^2 - X^{-2}q^{-2} + X^2q^4 - X^{-2}q^{-4} \right\} \quad (4)$$

The LHS of (4) returns:

$$\begin{aligned} & \frac{1}{z_1^2} \left\{ (-X^2q^2 + 2 - X^{-2}q^{-2}) + X^2q^4 - 2 + X^{-2}q^{-4} \right\} \\ &= \frac{1}{z_1^2} \left\{ X^2q^2(q^2 - 1) - X^{-2}q^{-2}(1 - q^2) \right\} \\ &= \frac{1}{z_1^2} \left\{ X^2q^2q(q - q^{-1}) - X^{-2}q^{-2}q^{-1}(q - q^{-1}) \right\} \\ &= \frac{1}{z_1} \left\{ X^2q^3 - X^{-2}q^{-3} \right\} = [2n+3]_q^\sim . \end{aligned} \quad (5)$$

The RHS of (4) yields:

$$\begin{aligned}
& \frac{1}{z_2} \left\{ X^2 q^2 (1 + q^2) - X^{-2} q^{-2} (1 + q^{-2}) \right\} \\
&= \frac{1}{z_2} \left\{ X^2 q^2 q (q^{-1} + q) - X^{-2} q^{-2} q^{-1} (q + q^{-1}) \right\} \\
&= \frac{[2]_q^\sim}{z_2} \left\{ X^2 q^3 - X^{-2} q^{-3} \right\} = \frac{1}{z_1} \left\{ X^2 q^3 - X^{-2} q^{-3} \right\} = [2n + 3]_q^\sim
\end{aligned}$$

which is the same as (5). We used in the Proof the obvious relation

$$[2]_q^\sim = \frac{z_2}{z_1}.$$