

A binomial form for the sum of consecutive quantum integers

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The classical formula stating that the sum of n consecutive numbers is equal to $(n + 1)$ choose n is quantized.

The classical formula $1 + 2 + \dots + n = \binom{n+1}{2}$ is quantized.

Set

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (1)$$

The quantized identity is then:

$$\sum_{i=0}^n [i]_q^\sim \frac{[2]_q^\sim{}^i}{[2]_q^\sim} = \left[\begin{matrix} n+1 \\ 2 \end{matrix} \right]_q^\sim, \quad n \in \mathbb{Z}_{>0}, \quad (2)$$

where

$$\left[\begin{matrix} n \\ 2 \end{matrix} \right]_q^\sim = \frac{[n]_q^\sim [n-1]_q^\sim}{[2]_q^\sim}.$$

For $n = 0, 1$, formula (2) is obviously true, because $\left[\begin{matrix} 2 \\ 2 \end{matrix} \right] = 1$. We next use induction on n ; formula (2) results from

$$[n+1]_q^\sim \frac{[2]_q^\sim{}^{n+1}}{[2]_q^\sim} \stackrel{?}{=} \frac{[n+1]_q^\sim}{[2]_q^\sim} [n+2]_q^\sim - [n]_q^\sim,$$

which is true because

$$[x+2]_q^\sim - [x]_q^\sim = [2]_q^\sim{}^{x+1}. \quad (3)$$

The latter formula is true because, with $z = q - q^{-1}$, the LHS of (3) is:

$$\begin{aligned} & \frac{1}{z} [q^{x+2} - q^{-2-x} - q^x + q^{-x}] = \\ & = \frac{1}{z} \begin{bmatrix} q^x & q^2 - 1 & + q^{-x} & 1 - q^2 \end{bmatrix} = \frac{1}{z} [q^x qz + q^{-x} q^{-1} z] \\ & = q^{x+1} + q^{-x-1} = [2]_q^\sim{}^{x+1}, \end{aligned}$$

because

$$[2]_q^\sim = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}.$$