

Rotor imbalance from kinetic energy: A temporal SE(3) formulation

Thad S. Morton

Rotor imbalance is derived from kinetic energy without prescribing centrifugal forces. A temporal SE(3) velocity generator produces a single kinetic energy integral whose expansion reveals the zeroth, first, and second mass moments governing translation, translation-rotation coupling, and rotation. These moments form a homogeneous mass-moment matrix from which the conventional inertia matrix and parallel-axis theorem are recovered. Differentiation of kinetic energy with respect to the temporal generator yields a unified momentum wrench containing linear and matrix angular momenta. Applications to an eccentric rigid rotor and a flexible-shaft Jeffcott rotor demonstrate how rotating forces, moments, and shaft whirl follow from the underlying mass distribution, including rotor bore offsets, eccentric masses, and misaligned axes. The formulation connects rotor imbalance and shaft response without requiring the manual gathering of relevant terms.

1. Introduction

This paper extends a previously developed SE(3) elasticity formulation (Morton 2026) to the dynamics of rotating machinery. Imbalance in rotating machinery can cause vibrations and even mechanical failure through cyclic stresses associated with rotor vibration. For pure rotation of a rotor about a fixed point, the kinetic energy is given by

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\ell} \cdot \boldsymbol{\omega}, \quad (1)$$

where $\boldsymbol{\ell}$ is the angular momentum vector, and $\boldsymbol{\omega}$ is the angular velocity vector. This paper identifies the energetic quantity whose gradient gives rotational momentum, translational momentum, and their coupling when the mass distribution is not centered on the axis of rotation. The rich machinery of the Lie group SE(3) and its Lie algebra $\mathfrak{se}(3)$ allows this to be done using essentially the same formulation developed for elasticity (Morton 2026).

The kinetic energy of a rigid rotor is conventionally expressed in terms of angular velocity and the inertia matrix, while rotor vibration is commonly described by equations of motion in generalized coordinates. The classical Jeffcott rotor (Jeffcott 1919) established a basic model for the lateral response of a flexible rotating shaft to mass imbalance. Nelson and McVaugh (1976) extended rotor-bearing analysis using finite elements incorporating rotary inertia, gyroscopic moments, axial loading, and variable-cross-section rotor elements. Genta (2005) treats elementary and finite-element models of rotating systems, and Friswell *et al.* (2010) develop increasingly detailed rotor models using generalized coordinates. The mass matrices used in these works describe inertia in the chosen generalized coordinates and are distinct from the geometric mass-moment matrices used in the present study.

Uicker (1965, 1967) introduced a 4×4 mass distribution matrix incorporating the zeroth, first, and second mass moments, represented herein by m_0 , $\mathbf{m}_1$, and $\mathbf{m}_2$, and employed it in kinetic energy calculations. Related homogeneous formulations were subsequently used in robot dynamics by Paul (1981) and Fu *et al.* (1987), including trace-based expressions for

kinetic energy. The homogeneous mass-moment matrix $\mathbf{J}$ derived herein is equivalent to Uicker’s matrix, apart from coordinate ordering: Uicker placed the scalar mass in the (1,1) position, whereas the present formulation places it in the (4,4) position to remain consistent with the SE(3) representation.

Unlike the classical rotational inertia matrix, the homogeneous mass-moment matrix retains all three levels of mass moments explicitly. As will be shown in §4.3, the conventional rotational inertia matrix follows from the second mass moment through $\mathbf{I} = \text{tr}(\mathbf{m}_2)\mathbf{1} - \mathbf{m}_2$. The geometric mass moments describe the underlying mass distribution independently of the generalized coordinates used to model structural motion. Thus added or missing material, displaced bores, manufacturing asymmetry, and non-coincident shaft and principal axes can be represented consistently through changes in the same mass distribution.

Another line of development concerned the unified representation of translational and rotational dynamics. Wittenburg (1977, Eq. 6.29) formulated coupled linear and angular impulse-velocity-increment relationships using a symmetric $6n \times 6n$ matrix for multibody systems. Featherstone (1983, 1987, 2008) subsequently developed a dynamics of what he referred to as “spatial vectors,” representing angular and linear quantities by six-component vectors and relating generalized velocity to momentum through a six-component “spatial velocity” vector and a 6×6 “spatial inertia” matrix. His formulation also provided an algebra for spatial motion and force vectors and efficient recursive algorithms. Although Wittenburg’s impulse-velocity operator differs from Featherstone’s spatial inertia, both demonstrate the advantages of combining translational and rotational dynamics in a unified matrix representation. Balafoutis *et al.* (1988) developed a related Cartesian-tensor formulation of robot dynamics using skew-symmetric matrices to represent vector cross products and simplify Newton–Euler calculations.

Lie-group methods provided a complementary geometric framework. Murray *et al.* (1994) presented rigid-body kinematics and mechanics using SE(3), twists, and wrenches. Park *et al.* (1995) formulated robot dynamics using Lie-algebra operations and a generalized 6×6 inertia matrix that includes translation-rotation coupling. Park and Kim (2000) treated kinetic energy as a left-invariant Riemannian metric on SE(3) and extended the formulation to coupled rigid bodies. These approaches organize rigid-body inertia through operators on generalized velocities, rather than directly through the homogeneous matrix of zeroth, first, and second mass moments.

Wensing *et al.* (2018) expressed kinetic energy directly in terms of the 4×4 pseudo-inertia matrix equivalent to Uicker’s mass distribution matrix and an SE(3) Lie-algebra velocity generator, and established physical-consistency conditions on inertial parameters. Laus and Selig (2020) represented point-mass pseudo-inertia through outer products of homogeneous position vectors, demonstrated its congruence transformation under rigid displacement, and investigated equimomental systems of point masses. The mass-moment matrix derived herein obeys the same congruence transformation.

Conventional Jeffcott rotor models generally introduce a rotating imbalance force $m\epsilon\Omega^2$ as an excitation in the equations of motion (see, e.g., Haslam *et al.* 2020). More general rotor models can represent moment imbalance and gyroscopic effects. Ziese *et al.* (2021) acknowledged possible unbalance moments arising from disk misalignment, but did not measure them because only single-plane balancing was performed. Such moments lie outside the classical two-degree-of-freedom Jeffcott model, although they can be included in more

general rotor-dynamic formulations.

The contribution of the present study is therefore not the introduction of mass moments, spatial inertia, translation-rotation coupling, or the homogeneous kinetic-energy trace formulation, all of which have established precedents. Rather, the formulation develops the zeroth, first, and second mass moments explicitly from the temporal SE(3) kinetic-energy integral, then uses its Lie-algebra variation to obtain linear and matrix angular momenta within a unified momentum wrench. Applied to rigid and flexible-shaft rotors, this construction connects mass-distribution asymmetry directly to rotating force and moment excitation and the resulting shaft response, without independently prescribing centrifugal forces. The conventional inertia matrix and parallel-axis theorem follow from the same mass moments, while their homogeneous representation preserves the connection between rigid-body kinematics, inertial excitation, and elastic response.

Nomenclature

b	bore offset relative to the rotor centroid [m] (57)
β	shaft bending angle [radians] (98)
c	vector from the shaft attachment point to center of mass [m] (79)-(81)
e	location of eccentric mass relative to point of attachment to shaft [m] (60)
J	mass moment matrix (4×4) (22), (23)
θ_i	rotational field/ orientation [radians]
$\Theta(z)$	rotation matrix (operator) [dimensionless]
L	angular momentum [N·m·s] (30)
m_0	mass [kg] (17)
m₁	mass first moment vector [kg·m] (18)
m₂	mass second moment matrix [kg·m ²] (19)
M	force moment (torque) [N·m] (73)
p	momentum [kg·m/s] (31)
r	position relative to center of mass [m]
R = r + c	position relative to the point of attachment to the shaft [m] (51)
$\tilde{\mathbf{R}}$	homogeneous representation of position R [m dimensionless] (7)
<i>T</i>	kinetic energy [J] (12), (122)
v	translation velocity [m/s]
V	total physical velocity (translation and rotation) [m/s] (9)
Ω	rotation rate matrix [rad/s] (4)
ω	rotation rate vector [rad/s] (47)
$\mathbf{X} = \begin{bmatrix} \Theta & \mathbf{a} \\ 0 & 1 \end{bmatrix}$	configuration map [m/m m] (2)
$\xi_0 = \begin{bmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{bmatrix}$	temporal SE(3) velocity generator [1/s m/s] (6)
$\zeta(t)$	displacement of the rotor allowed by a flexible shaft [m] (77)

2. Temporal SE(3) Formulation

As done for elasticity (Morton 2026), the configuration is defined as

$$\mathbf{X}(t) = \begin{bmatrix} \boldsymbol{\Theta}(t) & \mathbf{a}(t) \\ 0 & 1 \end{bmatrix} \in \text{SE}(3). \quad (2)$$

The time derivative is

$$\dot{\mathbf{X}}(t) = \begin{bmatrix} \dot{\boldsymbol{\Theta}}(t) & \dot{\mathbf{a}}(t) \\ 0 & 0 \end{bmatrix}.$$

However, this is not the body velocity. To obtain it, first use the temporal Maurer-Cartan form: $\boldsymbol{\xi}_0 = \mathbf{X}^{-1}\partial_0\mathbf{X}$, and

$$\mathbf{X}^{-1} = \begin{bmatrix} \boldsymbol{\Theta}^T & -\boldsymbol{\Theta}^T\mathbf{a} \\ 0 & 1 \end{bmatrix}.$$

Multiplying the two together, we obtain the temporal Lie-algebra field

$$\boldsymbol{\xi}_0 = \mathbf{X}^{-1}\partial_0\mathbf{X} = \begin{bmatrix} \boldsymbol{\Theta}^T\dot{\boldsymbol{\Theta}} & \boldsymbol{\Theta}^T\dot{\mathbf{a}} \\ 0 & 0 \end{bmatrix} \quad (3)$$

The subscript 0 denotes the temporal coordinate, distinguishing $\boldsymbol{\xi}_0$ from the spatial generators $\boldsymbol{\xi}_i$ used in the elasticity formulation (Morton 2026); therefore, $\partial_0 = \partial/\partial t$.

So angular velocity is

$$\boldsymbol{\Omega} = \boldsymbol{\Theta}^T\dot{\boldsymbol{\Theta}} = \boldsymbol{\Theta}^{-1}\dot{\boldsymbol{\Theta}} \in \mathfrak{so}(3), \quad (4)$$

and translational velocity is

$$\mathbf{v} = \dot{\boldsymbol{\Theta}}^{-1}\dot{\mathbf{a}}. \quad (5)$$

Therefore,

$$\boldsymbol{\xi}_0 = \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{v} \\ 0 & 0 \end{bmatrix}. \quad (6)$$

For a material point, the homogeneous coordinates are given by

$$\tilde{\mathbf{R}} = \begin{bmatrix} \mathbf{R} \\ 1 \end{bmatrix}, \quad (7)$$

its transformation law from one point $\tilde{\mathbf{R}}$ to some other point $\tilde{\mathbf{R}}'$ is

$$\tilde{\mathbf{R}}' = \mathbf{X}\tilde{\mathbf{R}}, \quad (8)$$

and the local velocity of the body is then

$$\tilde{\mathbf{V}} = \boldsymbol{\xi}_0\tilde{\mathbf{R}}, \quad \mathbf{V} = \boldsymbol{\Omega}\mathbf{R} + \mathbf{v}. \quad (9)$$

This is the temporal analogue of the elastic strain introduced previously (Morton 2026):

$$\boldsymbol{\eta}_i = \boldsymbol{\xi}_i\mathbf{R} = \boldsymbol{\Gamma}_{,i}\mathbf{R} + \boldsymbol{\epsilon}_{,i}$$

where

$$\boldsymbol{\xi}_i = \mathbf{X}^{-1}\partial_i\mathbf{X} = \begin{bmatrix} \boldsymbol{\Gamma}_{,i} & \boldsymbol{\epsilon}_{,i} \\ 0 & 0 \end{bmatrix}, \quad (10)$$

analogous to (6) (Morton 2026).

Expanding the velocity in (9) gives

$$\tilde{\mathbf{V}} = \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega}\mathbf{R} + \mathbf{v} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{V} \\ 0 \end{bmatrix}. \quad (11)$$

3. Kinetic Energy

Rather than beginning separately with a translating mass, a classical rotational inertia matrix, and coupling terms, the kinetic energy is simply defined by

$$T = \frac{1}{2} \int \rho \mathbf{V}^T \mathbf{V} d\mathcal{V}, \quad (12)$$

where $\mathbf{V}$ is given by (9). Substitute (9) into (12) to obtain

$$\begin{aligned} T &= \frac{1}{2} \int \rho (\boldsymbol{\Omega}\mathbf{R} + \mathbf{v})^T (\boldsymbol{\Omega}\mathbf{R} + \mathbf{v}) d\mathcal{V} \\ &= \frac{1}{2} \int \rho [(\boldsymbol{\Omega}\mathbf{R})^T (\boldsymbol{\Omega}\mathbf{R}) + \mathbf{R}^T \boldsymbol{\Omega}^T \mathbf{v} + \mathbf{v}^T \boldsymbol{\Omega}\mathbf{R} + \mathbf{v}^T \mathbf{v}] d\mathcal{V} \end{aligned}$$

The two middle terms are scalars and are transposes of each other:

$$\mathbf{R}^T \boldsymbol{\Omega}^T \mathbf{v} = (\mathbf{v}^T \boldsymbol{\Omega}\mathbf{R})^T = \mathbf{v}^T \boldsymbol{\Omega}\mathbf{R},$$

so they can be combined:

$$\begin{aligned} T &= \frac{1}{2} \int \rho [(\boldsymbol{\Omega}\mathbf{R})^T (\boldsymbol{\Omega}\mathbf{R}) + 2\mathbf{v}^T \boldsymbol{\Omega}\mathbf{R} + \mathbf{v}^T \mathbf{v}] d\mathcal{V} \\ &= \frac{1}{2} \int \rho \mathbf{R}^T \boldsymbol{\Omega}^T (\boldsymbol{\Omega}\mathbf{R}) d\mathcal{V} + \int \rho \mathbf{v}^T \boldsymbol{\Omega}\mathbf{R} d\mathcal{V} + \frac{1}{2} \int \rho \mathbf{v}^T \mathbf{v} d\mathcal{V} \end{aligned} \quad (13)$$

We can identify the following quantities:

1. Pure rotational kinetic energy contribution about the reference origin [this is (1)]:

$$T_{\text{rot}} = \frac{1}{2} \int \rho \mathbf{R}^T \boldsymbol{\Omega}^T (\boldsymbol{\Omega}\mathbf{R}) d\mathcal{V} \quad (14)$$

2. Translational kinetic energy contribution of the reference origin:

$$T_{\text{tran}} = \frac{1}{2} \int \rho \mathbf{v}^T \mathbf{v} d\mathcal{V} \quad (15)$$

3. General 3D translation-rotation coupling (imbalance):

$$T_{\text{cpl}} = \int \rho \mathbf{v}^T \boldsymbol{\Omega}\mathbf{R} d\mathcal{V} \quad (16)$$

The radial center-of-mass offset (to be denoted by $\mathbf{c}_\perp$) produces centrifugal imbalance due to lateral force and rotating whirl. The first and second mass moments capture the force and moment effects associated with rotor imbalance.

3.1. Mass Moments

Now define the following mass moments:

$$m_0 = \int \rho d\mathcal{V} \quad (\text{zeroth mass moment}) \quad (17)$$

$$\mathbf{m}_1 = \int \rho \mathbf{R} d\mathcal{V} \quad (\text{first mass moment}) \quad (18)$$

$$\mathbf{m}_2 = \int \rho \mathbf{R} \mathbf{R}^T d\mathcal{V} \quad (\text{second mass moment matrix}) \quad (19)$$

So, for example, the last relation expands as

$$\mathbf{m}_2 = \begin{bmatrix} \int \rho x^2 d\mathcal{V} & \int \rho xy d\mathcal{V} & \int \rho xz d\mathcal{V} \\ \int \rho yx d\mathcal{V} & \int \rho y^2 d\mathcal{V} & \int \rho yz d\mathcal{V} \\ \int \rho zx d\mathcal{V} & \int \rho zy d\mathcal{V} & \int \rho z^2 d\mathcal{V} \end{bmatrix}. \quad (20)$$

Using these moments, we can write the energy in (13) as

$$T = \frac{1}{2} \boldsymbol{\Omega} : \mathbf{m}_2 \boldsymbol{\Omega} + \mathbf{v}^T \boldsymbol{\Omega} \mathbf{m}_1 + \frac{1}{2} m_0 \mathbf{v}^T \mathbf{v}. \quad (21)$$

The three rigid-body quantities—rotational energy, translation-rotation coupling energy, and translational energy—were not introduced separately but instead arise from the same energy integral.

3.2. Homogeneous Mass-Moment Matrix

Using (7), we can write the entire homogeneous mass-moment matrix as simply:

$$\mathbf{J} = \int \rho \tilde{\mathbf{R}} \tilde{\mathbf{R}}^T d\mathcal{V}. \quad (22)$$

Expansion of this single expression produces the zeroth, first, and second mass moments naturally. The resulting energy contains a rotational term governed by $\mathbf{m}_2$, a translation-rotation coupling term governed by $\mathbf{m}_1$, and a translational term governed by m_0 . Thus the distinction between balanced and imbalanced mass distributions enters through the same hierarchy of geometric moments.

The matrix $\mathbf{J}$ can be written in terms of the mass moment matrices as:

$$\mathbf{J} = \int \rho \begin{bmatrix} \mathbf{R} \\ 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}^T & 1 \end{bmatrix} d\mathcal{V} = \begin{bmatrix} \mathbf{m}_2 & \mathbf{m}_1 \\ \mathbf{m}_1^T & m_0 \end{bmatrix} \quad (23)$$

Equation (23) is, up to the ordering of the homogeneous coordinate, the 4×4 mass-distribution matrix introduced by Uicker (1965, 1967) and is identical in form to the pseudo-inertia matrix used by Fu *et al.* (1987) and Wensing *et al.* (2018). Wensing's primary contribution was to establish physical consistency conditions on inertial parameters using this matrix. The present work provides an explicit derivation of its constituent moments from the temporal SE(3) kinetic-energy integral and their application to rotor imbalance and vibration.

3.3. Transformation of Mass Moments

In a frame transformed by (8), the mass-moment matrix becomes

$$\mathbf{J}' = \int \rho (\mathbf{X} \tilde{\mathbf{R}}) (\mathbf{X} \tilde{\mathbf{R}})^T d\mathcal{V}.$$

Since $\mathbf{X}$ is independent of the integration variable, we can pull both instances of it out of the integral (Laus and Selig 2020)

$$\mathbf{J}' = \mathbf{X} \left(\int \rho \tilde{\mathbf{R}} \tilde{\mathbf{R}}^T d\mathcal{V} \right) \mathbf{X}^T$$

$$\mathbf{J}' = \mathbf{X} \mathbf{J} \mathbf{X}^T \quad (24)$$

Therefore, a translation that looks messy in separate m_0 , $\mathbf{m}_1$, and $\mathbf{m}_2$ terms becomes a single congruence transformation. Translation of the reference point thereby changes the first and second moments simultaneously.

A congruence transformation is of the form

$$\mathbf{B} = \mathbf{H} \mathbf{A} \mathbf{H}^T,$$

whereas a similarity transformation is of the form

$$\mathbf{B} = \mathbf{H} \mathbf{A} \mathbf{H}^{-1}.$$

They coincide when $\mathbf{H}$ is orthogonal because then $\mathbf{H}^{-1} = \mathbf{H}^T$. (So, for example, the 3×3 rotation matrix $\mathbf{Q}$ in (71) below is orthogonal, so that transformation could be viewed either way.) But the homogeneous SE(3) matrix in (2) is not generally orthogonal when $\mathbf{a} \neq 0$. So the transformation of the mass moment matrix in (24) is a congruence transformation.

Incidentally, the ordinary transformation of a rank-two contravariant tensor is also a congruence transformation, with the coordinate Jacobian as its transformation matrix. The spatial Jacobian of an affine transformation retains the linear part, including rotation or deformation, but eliminates any constant term associated with an offset from the origin. Homogeneous coordinates retain that term, allowing (24) to transform the zeroth, first, and second mass moments together.

4. The Momentum Wrench

As in the elasticity formulation (Morton 2026), to obtain the wrench, take the derivative of energy with respect to ξ . The main difference from the elasticity study is that instead of obtaining the *force and moment* wrench by differentiating *strain* energy with respect to ξ_i given in (10), we obtain the *momentum* wrench by differentiating the *kinetic* energy with respect to ξ_0 , given in (6):

$$\mathbf{\Pi}_0 = \frac{\partial T}{\partial \xi_0}. \quad (25)$$

It is important that we do not differentiate with respect to the physical velocity field $\mathbf{V}$, but rather with respect to ξ_0 , the generalized Lie-algebra $\mathfrak{se}(3)$ velocity that generates this $\mathbf{V}$ field. This allows $\mathbf{\Pi}_0$ to represent both linear and angular momentum. The derivative in (25) is defined as a derivative on the Lie algebra $\mathfrak{se}(3)$ and its dual, to which its conjugate momentum belongs.

For skew-symmetric matrices representing three-dimensional vectors, we can start with the identity (Wittenburg 1977, p. 16, Eq. 1.28)

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} = \mathbf{b}\mathbf{a}^T - (\mathbf{a}^T\mathbf{b})\mathbf{1}.$$

Take the trace of both sides:

$$\begin{aligned} \text{tr}(\tilde{\mathbf{a}}\tilde{\mathbf{b}}) &= \text{tr}(\mathbf{b}\mathbf{a}^T) - 3\mathbf{a}^T\mathbf{b} = \mathbf{a}^T\mathbf{b} - 3\mathbf{a}^T\mathbf{b} \\ &= -2\mathbf{a}^T\mathbf{b}. \end{aligned}$$

Since skew-symmetric matrices satisfy $\tilde{\mathbf{a}}^T = -\tilde{\mathbf{a}}$, we have

$$\frac{1}{2} \text{tr}([\mathbf{a}]_{\times}^T [\mathbf{b}]_{\times}) = \mathbf{a}^T \mathbf{b}. \quad (26)$$

Here, $[\mathbf{a}]_{\times}$ and $[\mathbf{b}]_{\times}$ denote the skew-symmetric matrices associated with the three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$. Forms of this identity can be found in Simmonds (1982, p. 24) and Balafoutis *et al.* (1988, Rule R6). It allows the conventional angular momentum and angular velocity dot product to be expressed entirely in matrix form.

Take the variation of $\mathbf{V}$ in (9)

$$\delta \mathbf{V} = \delta \boldsymbol{\Omega} \mathbf{R} + \delta \mathbf{v}.$$

Then the variation of kinetic energy in (12) becomes

$$\delta T = \int \rho \mathbf{V}^T \delta \boldsymbol{\Omega} \mathbf{R} d\mathcal{V} + \int \rho \mathbf{V}^T \delta \mathbf{v} d\mathcal{V}.$$

If we define

$$\mathbf{N} \equiv \int_{\mathcal{V}} \rho \mathbf{V} \mathbf{R}^T d\mathcal{V}, \quad \mathbf{p} = \int_{\mathcal{V}} \rho \mathbf{V} d\mathcal{V},$$

we can write

$$\delta T = \text{tr}(\mathbf{N}^T \delta \boldsymbol{\Omega}) + \mathbf{p}^T \delta \mathbf{v}. \quad (27)$$

Since $\delta \boldsymbol{\Omega}$ is skew-symmetric, only the skew-symmetric part of $\mathbf{N}$ contributes:

$$\text{tr}(\mathbf{N}^T \delta \boldsymbol{\Omega}) = \frac{1}{2} \text{tr}[(\mathbf{N} - \mathbf{N}^T)^T \delta \boldsymbol{\Omega}]. \quad (28)$$

The factor 1/2 accounts for the duplication of the three independent components in the skew-symmetric matrix representation, consistent with the identity in (26). Defining

$$\mathbf{L} = \mathbf{N} - \mathbf{N}^T = \int_{\mathcal{V}} \rho (\mathbf{V} \mathbf{R}^T - \mathbf{R} \mathbf{V}^T) d\mathcal{V}$$

allows us to use (28) in (27) to write the first variation of kinetic energy as

$$\delta T = \frac{1}{2} \text{tr}(\mathbf{L}^T \delta \boldsymbol{\Omega}) + \mathbf{p}^T \delta \mathbf{v}. \quad (29)$$

The half-trace pairing and the identification of rotational and translational momenta with the dual of the SE(3) Lie algebra are established in geometric mechanics (Holm 2011, Eq. 2.4.14, p. 51; §7.1, p. 170).

Using the specific form of the half-trace pairing in (29) and $\boldsymbol{\xi}_0$ from (6), the derivative in (25) yields the following two relations:

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\Omega}} \quad (30)$$

and

$$\mathbf{p} = \frac{\partial T}{\partial \mathbf{v}}. \quad (31)$$

These results can be presented as

$$\boldsymbol{\Pi}_0 = \begin{bmatrix} \mathbf{L} & \mathbf{p} \\ 0 & 0 \end{bmatrix}. \quad (32)$$

This pairing identifies the momentum wrench in (32) without first reducing the rotational generator or angular momentum to their axial vectors.

4.1. Linear Momentum

The linear component of momentum $\mathbf{p}$ is found by differentiating kinetic energy (21) with respect to $\mathbf{v}$ as specified in (31). This gives

$$\mathbf{p} = \mathbf{\Omega} \mathbf{m}_1 + m_0 \mathbf{v} = \mathbf{\Omega} \mathbf{m}_1 + m \mathbf{v} \quad (33)$$

so that ordinary translational momentum and the momentum generated by rotation about an origin that is separated from the center of mass appear in one relation. Note that this is equivalent to taking the derivative of (13) with respect to $\mathbf{v}$. The integral form is

$$\mathbf{p} = \int_{\mathcal{V}} \rho (\mathbf{\Omega} \mathbf{R} + \mathbf{v}) d\mathcal{V}. \quad (34)$$

4.2. Angular Momentum

To calculate the rotational component of momentum, we take the derivative of the kinetic energy in (21) (or (13)) specified in (30). Beginning with (14), the rotational kinetic energy:

$$T_{\text{rot}} = \frac{1}{2} \int \rho \mathbf{R}^T \mathbf{\Omega}^T \mathbf{\Omega} \mathbf{R} d\mathcal{V}. \quad (35)$$

The angular velocity $\mathbf{\Omega}$ matrix is skew-symmetric and given by

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Use the identity

$$\mathbf{R}^T \mathbf{\Omega}^T \mathbf{\Omega} \mathbf{R} = \text{tr}(\mathbf{R}^T \mathbf{\Omega}^T \mathbf{\Omega} \mathbf{R}). \quad (36)$$

in (35). The scalar velocity term is rewritten here using the trace operator in order to express the rotational energy in terms of the geometric properties of the mass distribution. This will allow the position dependence to be collected into the second mass moment matrix $\mathbf{m}_2$. For example, since (36) is a scalar, we can cyclically permute the trace and obtain:

$$\text{tr}(\mathbf{\Omega}^T \mathbf{R}^T \mathbf{R} \mathbf{\Omega})$$

and use this in the energy term in (35). Then, after moving the integral inside the trace, we have

$$T_{\text{rot}} = \frac{1}{2} \text{tr} \left[\mathbf{\Omega} \left(\int \rho \mathbf{R}^T \mathbf{R} d\mathcal{V} \right) \mathbf{\Omega}^T \right].$$

Since

$$\mathbf{m}_2 = \int \rho \mathbf{R} \mathbf{R}^T d\mathcal{V}, \quad (37)$$

the rotational kinetic energy term becomes

$$T_{\text{rot}} = \frac{1}{2} \text{tr}(\mathbf{\Omega} \mathbf{m}_2 \mathbf{\Omega}^T). \quad (38)$$

The differential is

$$\delta T_{\text{rot}} = \frac{1}{2} \delta \left[\text{tr}(\mathbf{\Omega} \mathbf{m}_2 \mathbf{\Omega}^T) \right]$$

$$\delta T_{\text{rot}} = \frac{1}{2} \text{tr}(\delta \mathbf{\Omega} \mathbf{m}_2 \mathbf{\Omega}^T) + \frac{1}{2} \text{tr}(\mathbf{\Omega} \mathbf{m}_2 \delta \mathbf{\Omega}^T) \quad (39)$$

The two terms arise because the rotational generator $\mathbf{\Omega}$ appears twice in the quadratic energy, and constrained variation acts on both occurrences. Since the angular velocity matrix satisfies $\mathbf{\Omega} \in \mathfrak{so}(3)$, the relation $\mathbf{\Omega}^T = -\mathbf{\Omega}$ holds, and allowable variations have the form $\delta \mathbf{\Omega} \in \mathfrak{so}(3)$. Then (39) can be written

$$\delta T_{\text{rot}} = -\frac{1}{2} \text{tr}(\delta \mathbf{\Omega} \mathbf{m}_2 \mathbf{\Omega}) - \frac{1}{2} \text{tr}(\mathbf{\Omega} \mathbf{m}_2 \delta \mathbf{\Omega}). \quad (40)$$

For compatible matrices (meaning matrices that may not necessarily be square individually but form a closed product whose result is square), we can cyclically permute them inside a trace, as follows:

$$\text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{BCA}) = \text{tr}(\mathbf{CAB}).$$

Therefore, (40) can be written

$$\begin{aligned} \delta T_{\text{rot}} &= -\frac{1}{2} \text{tr}(\delta \mathbf{\Omega} \mathbf{m}_2 \mathbf{\Omega}) - \frac{1}{2} \text{tr}(\delta \mathbf{\Omega} \mathbf{\Omega} \mathbf{m}_2) \\ &= -\frac{1}{2} \text{tr}[\delta \mathbf{\Omega} (\mathbf{m}_2 \mathbf{\Omega} + \mathbf{\Omega} \mathbf{m}_2)] \end{aligned} \quad (41)$$

Therefore,

$$\mathbf{L}_{\text{rot}} = \mathbf{m}_2 \mathbf{\Omega} + \mathbf{\Omega} \mathbf{m}_2 \quad (42)$$

A matrix rotational inertia operator of this form appeared in the well-known extension of the Euler top to $\text{SO}(n)$ given by Manakov (1976). (Manakov showed that the n -dimensional rigid-body equations can be written as a Lax pair and are integrable.)

The identity in (41) can be written in the more compact form:

$$\delta T_{\text{rot}} = -\text{tr}(\delta \mathbf{\Omega} \mathbf{m}_2 \mathbf{\Omega}) \quad (43)$$

since

$$\text{tr}(\delta \mathbf{\Omega} \mathbf{m}_2 \mathbf{\Omega}) = \text{tr}(\delta \mathbf{\Omega} \mathbf{\Omega} \mathbf{m}_2).$$

This is not because $\mathbf{m}_2$ and $\mathbf{\Omega}$ can simply be permuted, but because on the left term, we can use the trace properties

$$\text{tr}(\mathbf{A}) = \text{tr}(\mathbf{A}^T) \quad \text{and} \quad (\mathbf{ABC})^T = \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T,$$

along with $\mathbf{m}_2^T = \mathbf{m}_2$, $\mathbf{\Omega}^T = -\mathbf{\Omega}$, and $\delta \mathbf{\Omega}^T = -\delta \mathbf{\Omega}$, to obtain the term on the right. It is because both $\mathbf{\Omega}$ and $\delta \mathbf{\Omega}$ are skew-symmetric that the two terms contribute equally under the trace.

Equation (42) gives the rotational contribution to angular momentum. To obtain the complete angular momentum about the selected reference origin, the translation-rotation coupling term in (21) must also be differentiated with respect to $\mathbf{\Omega}$ according to (30). Under the skew-symmetric variation $\delta \mathbf{\Omega}$, this coupling term in (21) is

$$\delta T = \mathbf{v}^T \delta \mathbf{\Omega} \mathbf{m}_1. \quad (44)$$

The matrix $\delta \mathbf{\Omega}$ must be skew-symmetric ($\delta \mathbf{\Omega}^T = -\delta \mathbf{\Omega}$), so

$$\mathbf{v}^T \delta \mathbf{\Omega} \mathbf{m}_1 = -\mathbf{m}_1^T \delta \mathbf{\Omega} \mathbf{v}.$$

Therefore, we can rewrite (44) as

$$\delta T_{\text{cpl}} = \frac{1}{2}(\mathbf{v}^T \delta \boldsymbol{\Omega} \mathbf{m}_1 - \mathbf{m}_1^T \delta \boldsymbol{\Omega} \mathbf{v}) = \frac{1}{2} \text{tr}[(\mathbf{m}_1 \mathbf{v}^T - \mathbf{v} \mathbf{m}_1^T) \delta \boldsymbol{\Omega}]$$

$$\delta T_{\text{cpl}} = \frac{1}{2} \text{tr}[(\mathbf{v} \mathbf{m}_1^T - \mathbf{m}_1^T \mathbf{v})^T \delta \boldsymbol{\Omega}]$$

By comparing this with (29), we see that the coupling term contributes the matrix $\mathbf{L}_{\text{cpl}} = \mathbf{v} \mathbf{m}_1^T - \mathbf{m}_1^T \mathbf{v}$. Consequently, the complete angular momentum includes both this first-moment coupling contribution and the rotational second-moment contribution given in (42):

$$\mathbf{L} = \underbrace{\mathbf{m}_2 \boldsymbol{\Omega} + \boldsymbol{\Omega} \mathbf{m}_2}_{\mathbf{L}_{\text{rot}}} + \underbrace{\mathbf{v} \mathbf{m}_1^T - \mathbf{m}_1^T \mathbf{v}}_{\mathbf{L}_{\text{cpl}}}. \quad (45)$$

Incidentally, for a single particle, if we let

$$\mathbf{p} = m\mathbf{V}, \quad \mathbf{V} = \boldsymbol{\Omega} \mathbf{R} + \mathbf{v},$$

then the above relation can be written

$$\mathbf{L} = \mathbf{p} \mathbf{R}^T - \mathbf{R} \mathbf{p}^T. \quad (46)$$

This skew-symmetric matrix is consistent with the classical $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$. The apparent sign reversal comes from converting the cross product into a matrix:

$$[\mathbf{a} \times \mathbf{b}]_{\times} = \mathbf{b} \mathbf{a}^T - \mathbf{a} \mathbf{b}^T,$$

where $[\mathbf{a} \times \mathbf{b}]_{\times}$ denotes the skew-symmetric matrix that represents the cross product vector.

For an extended body, integration over the body gives

$$\mathbf{L} = \int_{\mathbb{V}} \rho (\mathbf{V} \mathbf{R}^T - \mathbf{R} \mathbf{V}^T) d\mathbb{V}.$$

4.3. Classical Moment of Inertia

If we let

$$\mathbf{n} = \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}$$

be the unit vector along the rotation axis, the scalar $\mathbf{n}^T \mathbf{R}$ is the component of $\mathbf{R}$ along the axis. Then, multiplying by $\mathbf{n}$ gives that component as a vector $\mathbf{n} \mathbf{n}^T \mathbf{R}$ parallel to the axis. So $\mathbf{n} \mathbf{n}^T$ extracts the axial part; therefore, $\mathbf{I} - \mathbf{n} \mathbf{n}^T$ removes the axial component. Consequently, the identity

$$\boldsymbol{\Omega}^T \boldsymbol{\Omega} = \|\boldsymbol{\omega}\|^2 (\mathbf{I} - \mathbf{n} \mathbf{n}^T) \quad (47)$$

says that the squared rotational generator (the left side) is a scaled orthogonal projection onto the two-dimensional subspace normal to the rotation axis. The fact that

$$\boldsymbol{\omega} = \|\boldsymbol{\omega}\| \mathbf{n}$$

can be used to write (47) as

$$\boldsymbol{\Omega}^T \boldsymbol{\Omega} = (\boldsymbol{\omega}^T \boldsymbol{\omega}) \mathbf{I} - \boldsymbol{\omega} \boldsymbol{\omega}^T.$$

Now post-multiply by the second mass moment $\mathbf{m}_2$, and take the trace to obtain

$$\begin{aligned}\mathrm{tr}(\boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{m}_2) &= \mathrm{tr}\{[(\boldsymbol{\omega}^T \boldsymbol{\omega}) \mathbf{1} - \boldsymbol{\omega} \boldsymbol{\omega}^T] \mathbf{m}_2\} \\ &= (\boldsymbol{\omega}^T \boldsymbol{\omega}) \mathrm{tr}(\mathbf{m}_2) - \mathrm{tr}(\boldsymbol{\omega} \boldsymbol{\omega}^T \mathbf{m}_2)\end{aligned}$$

By a cyclic (double) permutation,

$$\mathrm{tr}(\boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{m}_2) = (\boldsymbol{\omega}^T \boldsymbol{\omega}) \mathrm{tr}(\mathbf{m}_2) - \mathrm{tr}(\boldsymbol{\omega}^T \mathbf{m}_2 \boldsymbol{\omega}),$$

and because $\boldsymbol{\omega}^T \mathbf{m}_2 \boldsymbol{\omega}$ is a scalar,

$$\begin{aligned}\mathrm{tr}(\boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{m}_2) &= (\boldsymbol{\omega}^T \boldsymbol{\omega}) \mathrm{tr}(\mathbf{m}_2) - \boldsymbol{\omega}^T \mathbf{m}_2 \boldsymbol{\omega} \\ &= \boldsymbol{\omega}^T [\mathrm{tr}(\mathbf{m}_2) \mathbf{1} - \mathbf{m}_2] \boldsymbol{\omega}\end{aligned}\tag{48}$$

In the square brackets, we see that the classical moment of inertia matrix appears:

$$\mathbf{I} = \mathrm{tr}(\mathbf{m}_2) \mathbf{1} - \mathbf{m}_2.\tag{49}$$

Therefore, the rotational energy in (38) can be written in the classical form

$$\frac{1}{2} \mathrm{tr}(\boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{m}_2) = \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega}.\tag{50}$$

Therefore, the classical rotational inertia matrix is one part the mass-distribution operator generated by the rotational Lie algebra action. The vector form reveals the geometric decomposition that is hidden by the compact matrix operator; however, the matrix form preserves the operator structure.

Here, all three— m_0 , $\mathbf{m}_1$, and $\mathbf{m}_2$ —emerge directly by expanding the single SE(3) kinetic-energy integral generated by $\mathbf{V} = \boldsymbol{\Omega} \mathbf{R} + \mathbf{v}$. So they appear in the rotational, translation-rotation coupling, and translational energy terms without being introduced separately. Thus these mass moments preserve information about the underlying mass distribution and are more fundamental than the conventional inertia matrix, which only contains $\mathbf{m}_2$.

The rotational component of momentum may be written as the skew matrix shown in (42) and with additional first-moment terms when a translational velocity is present. The matrix representation is retained here rather than immediately replacing the angular velocity and angular momentum by their axial vectors, because the matrix representation keeps the rotational generator and its conjugate momentum within the Lie-algebra representation used by the SE(3) formulation.

4.4. Parallel-Axis Theorem

When the reference frame is placed at the center of mass, the first mass moment $\mathbf{m}_1$ vanishes, eliminating the translation-rotation coupling term in (21). The kinetic energy then separates into rotational and translational contributions. The conventional 6×6 spatial-inertia formulation of Featherstone (1987) provides an equivalent description, with separate angular and translational blocks when referenced to the center of mass. In the present homogeneous mass-moment formulation, the classical rotational inertia matrix was recovered in (49) directly from the second mass moment. The parallel-axis theorem will now be derived from the transformation of that moment when the reference origin is displaced from the center of mass.

The parallel-axis theorem results from shifting the reference origin away from the center-of-mass by a vector $\mathbf{c}$, as follows:

$$\mathbf{R} = \mathbf{r} + \mathbf{c}. \quad (51)$$

Here, $\mathbf{c}$ is the vector from the shaft attachment point to the center-of-mass, $\mathbf{r}$ is the variable measure from the center of mass to any point at which mass is distributed. Substitute (51) into (37) to obtain

$$\mathbf{m}_2 = \int (\mathbf{r} + \mathbf{c})(\mathbf{r} + \mathbf{c})^T dm.$$

and expand

$$\mathbf{m}_2 = \int \mathbf{r}\mathbf{r}^T dm + \int \mathbf{r}\mathbf{c}^T dm + \int \mathbf{c}\mathbf{r}^T dm + \int \mathbf{c}\mathbf{c}^T dm.$$

Here, the second mass moment about the center of mass is given by

$$\mathbf{m}_{2,C} = \int \mathbf{r}\mathbf{r}^T dm. \quad (52)$$

Since $\mathbf{r}$ is measured from the center of mass, we know that $\int \mathbf{r} dm = 0$. Therefore,

$$\int \mathbf{r}\mathbf{c}^T dm = 0 \quad \text{and} \quad \int \mathbf{c}\mathbf{r}^T dm = 0$$

This leaves

$$\mathbf{m}_2 = \int \mathbf{r}\mathbf{r}^T dm + m_0 \mathbf{c}\mathbf{c}^T,$$

or, by (52),

$$\mathbf{m}_2 = \mathbf{m}_{2,C} + m_0 \mathbf{c}\mathbf{c}^T. \quad (53)$$

This is the translation law for the second mass-moment matrix $\mathbf{m}_2$.

To obtain the classical parallel-axis theorem, substitute the above relation into the classical inertia matrix given in (49):

$$\begin{aligned} \mathbf{I} &= \text{tr}(\mathbf{m}_{2,C} + m_0 \mathbf{c}\mathbf{c}^T) \mathbf{1} - (\mathbf{m}_{2,C} + m_0 \mathbf{c}\mathbf{c}^T) \\ &= \text{tr}(\mathbf{m}_{2,C}) \mathbf{1} - \mathbf{m}_{2,C} + m_0 [\text{tr}(\mathbf{c}\mathbf{c}^T) \mathbf{1} - \mathbf{c}\mathbf{c}^T] \end{aligned}$$

Since

$$\text{tr}(\mathbf{c}\mathbf{c}^T) = \mathbf{c}\mathbf{c}^T = c^2,$$

we have

$$\mathbf{I} = \mathbf{I}_C + m_0 [(\mathbf{c}\mathbf{c}^T) \mathbf{1} - \mathbf{c}\mathbf{c}^T],$$

This recovers the classical parallel-axis theorem in matrix form (Wittenburg 1977, §3.3.1). Thus the parallel-axis relation follows from translation of the second mass moment $\mathbf{m}_2$ that emerged from the SE(3) kinetic-energy integral.

4.5. Mass-Moment Coupling in Rotor Dynamics

The trace formulation of kinetic energy retains the contributions of products of inertia associated with the prescribed rotational motion. In general three-dimensional rotation, these contributions need not vanish. However, for rotation about a fixed coordinate axis, some products of inertia may not contribute to the scalar kinetic energy even though they contribute to angular momentum and the resulting moments. For example, rotation about

the z -axis gives $T_{\text{rot}} = \Omega^2(\mathbf{m}_{xx} + \mathbf{m}_{yy})/2$, whereas the transverse angular momentum components depend on $\mathbf{m}_{xz}$ and $\mathbf{m}_{yz}$. Likewise, the first mass moment $\mathbf{m}_1$ governs translation-rotation coupling and imbalance-induced linear momentum. Consequently, the full mass-moment representation remains important even when particular terms do not appear explicitly in the scalar kinetic-energy expression.

For a perfectly balanced rotor aligned with principal axes,

$$m_{xy} = m_{yz} = m_{zx} = 0.$$

Then $\mathbf{m}_2$ is diagonal, and the trace expression becomes deceptively simple:

$$T_{\text{rot}} = \frac{1}{2}(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2).$$

When imbalance occurs, $\mathbf{m}_1 \neq 0$ and/or $\mathbf{m}_2$ has off-diagonal terms. Although the trace expression is compact, it is formulated in terms of the complete second mass-moment matrix, even though not all of its components necessarily contribute to the scalar kinetic energy for a given motion. Diagonalization of the conventional inertia matrix is a special operation associated with principal axes and can obscure the coupling terms that become relevant for imperfect rotors.

5. Applications

5.1. Example 1: Rigid Rotor with Off-Axis Mass Distribution

Consider a disk of radius $R_O = 0.2$ m and thickness $H = 0.03$ m, rotating at 1000 rpm. The rotor bore is misaligned (off-of-center) by 3 mm in the x direction as shown in Figure 1. Assume the mass of the disk is $m_d = \rho V = 3.7$ kg. In addition to this, there is a concentrated 0.2-kg mass located at a point on the rotor outer perimeter on a line perpendicular to the bore misalignment (in the $-y$ direction). The axial offset of this lumped mass relative to the point of attachment of the rotor to the shaft is $e_3 = 0.05$ m. Also, the disk is mounted with its principal axis tilted by $\alpha = 30^\circ$. Determine the bearing forces and the resulting moments.

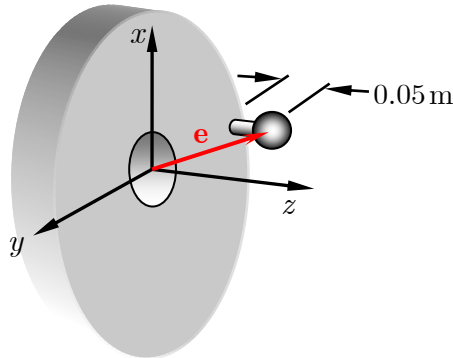


Figure 1. Unbalanced rotor with an eccentric lumped mass.

Rotation is about the z -axis, so the skew-symmetric angular velocity matrix is

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -104.7 & 0 \\ 104.7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (54)$$

5.1.1 Mass Moments

Second Mass Moment

Start with (52) for the body-fixed second mass moment of the disk

$$\mathbf{m}_{2,C}^{(d)} = \int \mathbf{r} \mathbf{r}^T dm. \quad (55)$$

Compute its components:

$$\begin{aligned} \int x^2 dm &= \int y^2 dm = \frac{1}{4} m_d R_O^2, \\ \int z^2 dm &= \frac{1}{12} m_d H^2, \end{aligned}$$

with

$$m_d = 3.7 \text{ kg}, \quad R_O = 0.2 \text{ m}, \quad H = 0.03 \text{ m}.$$

Substituting values gives

$$\int x^2 dm = \frac{1}{4} m_d R_O^2 = \frac{1}{4} (3.7)(0.2)^2 = 0.0370 \text{ kg m}^2$$

and

$$\int z^2 dm = \frac{1}{12} m_d H^2 = \frac{1}{12} (3.7)(0.03)^2 = 0.0002775 \text{ kg m}^2.$$

Therefore, the body-fixed second mass moment of the disk is

$$\mathbf{m}_{2,C}^{(d)} = \begin{bmatrix} 0.0370 & 0 & 0 \\ 0 & 0.0370 & 0 \\ 0 & 0 & 0.0002775 \end{bmatrix} \text{ kg m}^2. \quad (56)$$

This is the distributed disk contribution that will be used in a relation like (53).

Now account for the offset of the bore. Since the hole is shifted in the positive x direction, its eccentric mass is effectively located at

$$\mathbf{b} = \begin{bmatrix} -0.003 \\ 0 \\ 0 \end{bmatrix} \text{ m} \quad (57)$$

relative to the point of attachment of the rotor to the shaft. Since the eccentricity arises from the displacement of the bore, it is more difficult to illustrate in Figure 1. Applying the second-moment translation to the bore offset $\mathbf{b}$ gives that offset contribution to $\mathbf{m}_2$:

$$\begin{aligned}
m_d \mathbf{b} \mathbf{b}^T &= 3.7 \begin{bmatrix} (-0.003)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kg m}^2 \\
m_d \mathbf{b} \mathbf{b}^T &= \begin{bmatrix} 3.33 \times 10^{-5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kg m}^2
\end{aligned} \tag{58}$$

So, as done in (53), the total $\mathbf{m}_2$ for displaced disk alone is found by adding (56) and (58):

$$\begin{aligned}
\mathbf{m}_2^{(b)} &= \mathbf{m}_{2C}^{(d)} + m_d \mathbf{b} \mathbf{b}^T \\
&= \begin{bmatrix} 0.037033 & 0 & 0 \\ 0 & 0.037000 & 0 \\ 0 & 0 & 0.0002775 \end{bmatrix} \text{kg m}^2
\end{aligned} \tag{59}$$

Now do something similar for the lumped mass. The lumped mass lies on the disk perimeter measured from the disk center; but the disk center itself is at $x = -0.003$ m relative to the shaft, so the shaft-relative position of the eccentric lumped mass is

$$\mathbf{e} = \begin{bmatrix} -0.003 \\ -0.200 \\ 0.050 \end{bmatrix} \text{m} \tag{60}$$

relative to the point of attachment to the shaft.

Applying to the eccentric mass the same second-moment translation that produced the far-right term in (53) produces a term $m_e \mathbf{e} \mathbf{e}^T$. The outer product is

$$\mathbf{e} \mathbf{e}^T = \begin{bmatrix} (-0.003)^2 & (-0.003)(-0.2) & (-0.003)(0.05) \\ (-0.2)(-0.003) & (-0.2)^2 & (-0.2)(0.05) \\ (0.05)(-0.003) & (0.05)(-0.2) & (0.05)^2 \end{bmatrix} \text{m}^2.$$

Therefore, after multiplying by the lumped mass $m_e = 0.2 \text{ kg}$, we have

$$m_e \mathbf{e} \mathbf{e}^T = \begin{bmatrix} 0.0000018 & 0.00012 & -0.00003 \\ 0.00012 & 0.00800 & -0.00200 \\ -0.00003 & -0.00200 & 0.00050 \end{bmatrix} \text{kg m}^2. \tag{61}$$

The total second mass moment is then the sum of (59) and (61):

$$\mathbf{m}_2 = \mathbf{m}_2^{(b)} + m_e \mathbf{e} \mathbf{e}^T.$$

Substituting values gives

$$\begin{aligned}
\mathbf{m}_2 &= \begin{bmatrix} 0.037033 & 0 & 0 \\ 0 & 0.037000 & 0 \\ 0 & 0 & 0.0002775 \end{bmatrix} + \begin{bmatrix} 0.0000018 & 0.00012 & -0.00003 \\ 0.00012 & 0.00800 & -0.00200 \\ -0.00003 & -0.00200 & 0.00050 \end{bmatrix} \\
&= \begin{bmatrix} 0.0370351 & 0.000120 & -0.00003 \\ 0.000120 & 0.045000 & -0.00200 \\ -0.00003 & -0.00200 & 0.0007775 \end{bmatrix} \text{ kg m}^2
\end{aligned} \tag{62}$$

The rotor mass distribution is rotated by $\alpha = 30^\circ$ about the x -axis relative to the shaft, so the rotation matrix is

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix},$$

and the second mass moment in shaft coordinates becomes

$$\mathbf{m}_2^{\text{tilt}} = \mathbf{Q} \mathbf{m}_2 \mathbf{Q}^T,$$

which is

$$\mathbf{m}_2^{\text{tilt}} = \begin{bmatrix} 0.037035 & 1.1892 \times 10^{-4} & 3.4019 \times 10^{-5} \\ 1.1892 \times 10^{-4} & 0.035676 & 0.018149 \\ 3.4019 \times 10^{-5} & 0.018149 & 0.010101 \end{bmatrix} \text{ kg m}^2. \tag{63}$$

First Mass Moment

According to (18), the disk first moment (due to the misaligned bore) is

$$\mathbf{m}_1^{(b)} = m_d \mathbf{b} = 3.7 \begin{bmatrix} -0.003 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.0111 \\ 0 \\ 0 \end{bmatrix} \text{ kg m},$$

and that of the eccentric mass is

$$\mathbf{m}_1^{(e)} = m_e \mathbf{e} = 0.2 \begin{bmatrix} -0.003 \\ -0.200 \\ 0.050 \end{bmatrix}$$

or

$$\mathbf{m}_1^{(e)} = \begin{bmatrix} -0.0006 \\ -0.0400 \\ 0.0100 \end{bmatrix} \text{ kg m}$$

For the total $\mathbf{m}_1$, add the above mass-weighted first moments:

$$\begin{aligned} \mathbf{m}_1 &= \mathbf{m}_1^{(b)} + \mathbf{m}_1^{(e)} \\ &= \begin{bmatrix} -0.0111 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.0006 \\ -0.0400 \\ 0.0100 \end{bmatrix} = \begin{bmatrix} -0.0117 \\ -0.0400 \\ 0.0100 \end{bmatrix} \text{ kg m} \end{aligned} \quad (64)$$

To this first moment, apply the same tilt that was applied to the second moment:

$$\mathbf{m}_1^{\text{tilt}} = \mathbf{Q} \mathbf{m}_1 = \begin{bmatrix} -0.01170 \\ -0.03964 \\ -0.01134 \end{bmatrix} \text{ kg m}.$$

5.1.2 Kinetic Energy

The kinetic energy is given by (21). The shaft reference point is fixed, so $\mathbf{v} = 0$. Therefore, (21) becomes

$$T = \frac{1}{2} \text{tr}(\mathbf{\Omega} \mathbf{m}_2 \mathbf{\Omega}^T). \quad (65)$$

Since the rotor spins about the z -axis, this becomes

$$T = \frac{1}{2} \omega^2 (m_{xx} + m_{yy}),$$

Using (63)

$$m_{xx} = 0.037035 \text{ kg m}^2, \quad m_{yy} = 0.035676 \text{ kg m}^2$$

Therefore,

$$\begin{aligned} T &= \frac{1}{2} 104.72^2 (0.037035 + 0.035676) \\ &= 398.68 \text{ J} \end{aligned}$$

5.1.3 Linear Momentum

The linear momentum is found by (33)

$$\mathbf{p} = \mathbf{\Omega} \mathbf{m}_1^{\text{tilt}} + m \mathbf{v} = \begin{bmatrix} 0 & -104.7 & 0 \\ 104.7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.01170 \\ -0.03964 \\ -0.01134 \end{bmatrix} \text{ kg m/s} \quad (66)$$

Here, the matrix $\mathbf{\Omega}$ for angular velocity about the z -axis is given by (54). Therefore,

$$\mathbf{p} = \begin{bmatrix} 4.15 \\ -1.225 \\ 0 \end{bmatrix} \text{ kg m/s}$$

Although the momentum in (66) is constant in the body-fixed frame (denote it $\mathbf{p}^B$), its direction changes continuously in inertial space. So the linear momentum expressed in inertial coordinates is

$$\mathbf{p}^S = \mathbf{\Theta} \mathbf{p}^B, \quad (67)$$

and its time derivative is $\dot{\mathbf{p}}^S = \mathbf{\Theta} \dot{\mathbf{p}}^B + \dot{\mathbf{\Theta}} \mathbf{p}^B$. From (4), $\dot{\mathbf{\Theta}} = \mathbf{\Theta} \mathbf{\Omega}$, where the matrix $\mathbf{\Omega}$ is rotation of the body-fixed frame relative to inertial space. Therefore, (67) can be written

$$\dot{\mathbf{p}}^S = \mathbf{\Theta}(\dot{\mathbf{p}}^B + \mathbf{\Omega} \mathbf{p}^B).$$

For constant angular velocity and a fixed mass distribution, $\dot{\mathbf{p}}^B = 0$. Therefore, the force obtained from Newton's law $\mathbf{F}^S = \dot{\mathbf{p}}^S$ becomes $\mathbf{F}^B = \mathbf{\Omega} \mathbf{p}^B$ when expressed in body coordinates. The rotating centripetal force is then:

$$\begin{aligned} \mathbf{F} &= \mathbf{\Omega} \mathbf{p} \\ &= \begin{bmatrix} 0 & -104.7 & 0 \\ 104.7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4.15 \\ -1.225 \\ 0 \end{bmatrix} = \begin{bmatrix} 128.3 \\ 434.5 \\ 0 \end{bmatrix} \text{ N} \end{aligned} \quad (68)$$

This is the bearing force caused by the radial imbalance. So the force exerted by the rotor on the bearings would have the opposite sign. If the disk is located at the mid-span of the shaft, the resultant transverse force contributes equally to the two bearing reactions. The accompanying moment excitation produces additional, generally unequal reaction components.

Unlike conventional imbalance descriptions that treat force and moment imbalance separately, the present formulation shows that both effects arise from a single object describing the spatial distribution of mass through the first and second mass moments.

5.1.4 Angular Momentum

The angular momentum is given by (30):

$$\mathbf{L} = \frac{\partial T}{\partial \mathbf{\Omega}}.$$

As in the spatial formulation where the strain energy integral revealed the sectional geometric quantities governing elastic response, the kinetic-energy integral reveals the mass moments governing inertial response (Morton 2026).

The angular momentum is given by (42):

$$\mathbf{L}_{\text{rot}} = \mathbf{m}_2 \mathbf{\Omega} + \mathbf{\Omega} \mathbf{m}_2 \quad (69)$$

The term $\mathbf{m}_2 \mathbf{\Omega}$ is found multiplying (63) and (54):

$$\mathbf{m}_2^{\text{tilt}} \boldsymbol{\Omega} = \begin{bmatrix} 0.012451 & -3.87758 & 0 \\ 3.735191 & -0.01245 & 0 \\ 1.900190 & -0.003562 & 0 \end{bmatrix} \text{kg m}^2/\text{s}.$$

Likewise,

$$\boldsymbol{\Omega} \mathbf{m}_2^{\text{tilt}} = \begin{bmatrix} -0.01245 & -3.73532 & -1.90019 \\ 3.87758 & 0.012451 & 0.003562 \\ 0 & 0 & 0 \end{bmatrix} \text{kg m}^2/\text{s}.$$

Therefore, (69) gives

$$\mathbf{L}_{\text{rot}} = \begin{bmatrix} 0 & -7.61289 & -1.90019 \\ 7.61289 & 0 & 0.003562 \\ 1.90019 & -0.003562 & 0 \end{bmatrix} \text{kg m}^2/\text{s}. \quad (70)$$

And we see that $\mathbf{L}$ is skew-symmetric, as expected:

$$\mathbf{L}^T = -\mathbf{L},$$

with only three independent components.

This angular momentum is constant in the rotating frame (call it $\mathbf{L}^B$), but it is not constant in inertial space. So we transform it to inertial coordinates.

$$\mathbf{L}^S(t) = \boldsymbol{\Theta}(t) \mathbf{L}^B \boldsymbol{\Theta}^T(t) \quad (71)$$

This is a congruence transformation.

Differentiate both sides of (71) with respect to time

$$\dot{\mathbf{L}}^S(t) = \dot{\boldsymbol{\Theta}} \mathbf{L}^B \boldsymbol{\Theta}^T + \boldsymbol{\Theta} \dot{\mathbf{L}}^B \boldsymbol{\Theta}^T + \boldsymbol{\Theta} \mathbf{L}^B \dot{\boldsymbol{\Theta}}^T.$$

Then, because of (4), $\dot{\boldsymbol{\Theta}} = \boldsymbol{\Theta}^T \boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Theta}}^T = -\boldsymbol{\Omega} \boldsymbol{\Theta}^T$. So the above relation can be written

$$\dot{\mathbf{L}}^S = \boldsymbol{\Theta}(\dot{\mathbf{L}}^B + \boldsymbol{\Omega} \mathbf{L}^B - \mathbf{L}^B \boldsymbol{\Omega}) \boldsymbol{\Theta}^T. \quad (72)$$

Recall the classical governing equation

$$\dot{\mathbf{L}}^S = \mathbf{M}^S. \quad (73)$$

The right side of it can be transformed as follows: $\mathbf{M}^S = \boldsymbol{\Theta} \mathbf{M}^B \boldsymbol{\Theta}^T$. Therefore,

$$\dot{\mathbf{L}}^S = \boldsymbol{\Theta} \mathbf{M}^B \boldsymbol{\Theta}^T. \quad (74)$$

By comparing (72) and (74), we can make the following identification:

$$\mathbf{M}^B = (\dot{\mathbf{L}}^B + \boldsymbol{\Omega} \mathbf{L}^B - \mathbf{L}^B \boldsymbol{\Omega}).$$

In body coordinates, $\mathbf{L}^B$ is constant, so $\dot{\mathbf{L}}^B = 0$. And

$$\mathbf{M}^B = \boldsymbol{\Omega} \mathbf{L}^B - \mathbf{L}^B \boldsymbol{\Omega} \quad (75)$$

Therefore, the moment comes from the commutator, not necessarily from an ordinary componentwise time derivative of $\mathbf{L}^B$:

$$\mathbf{M}^B = [\boldsymbol{\Omega}, \mathbf{L}^B].$$

Compute the terms needed for the right side of (75):

$$\boldsymbol{\Omega} \mathbf{L}^B = \begin{bmatrix} -797.064 & 0 & -0.37292 \\ 0 & -797.064 & -198.951 \\ 0 & 0 & 0 \end{bmatrix} \text{ N m}$$

The product in reverse order is

$$\mathbf{L}^B \boldsymbol{\Omega} = \begin{bmatrix} -797.064 & 0 & 0 \\ 0 & -797.064 & 0 \\ -0.37292 & -198.951 & 0 \end{bmatrix} \text{ N m}$$

Therefore, (75) becomes

$$\mathbf{M}^B = \begin{bmatrix} 0 & 0 & -0.37292 \\ 0 & 0 & -198.951 \\ 0.37292 & 198.951 & 0 \end{bmatrix} \text{ N m}$$

All diagonal elements cancel so that the resulting moment matrix is skew-symmetric.

This example assumed $\mathbf{v} = 0$, corresponding to a rotor whose reference point is fixed. If the rotor translates, as in the next example, the linear momentum will remain

$$\mathbf{p} = \boldsymbol{\Omega} \mathbf{m}_1 + m_0 \mathbf{v} \quad (76)$$

instead of simplifying to (66). The first term on the right represents imbalance-induced momentum due to rotation about a non-central axis, while the second term is the ordinary translational momentum. Both contributions will arise from the same kinetic-energy gradient.

5.2. Example 2: Flexible Shaft (Jeffcott Rotor)

Now let the shaft be flexible so that whirling of the rotor can occur as shown in Figure 2. Again, disk radius is $R_O = 0.2$ m, and thickness is $H = 0.03$ m. It now rotates at 4000 rpm. Assume the mass of the disk is again $m_d = \rho V = 3.7$ kg, and the concentrated 0.2-kg mass is again located at a point on the rotor outer perimeter on a line perpendicular to the bore misalignment (in the $-y$ direction). The shaft has diameter 0.06 m, length 0.5 m, and Young's modulus $E = 200$ GPa. If the coordinate origin is placed on a line connecting the rigid bearings, the axial view looks like that in Figure 3.

Displacement of the rotor allowed by the flexible shaft will be denoted by $\boldsymbol{\zeta}$:

$$\boldsymbol{\zeta}(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ 0 \end{bmatrix} \quad (77)$$

This is the same displacement notation used in the elasticity study (Morton 2026).

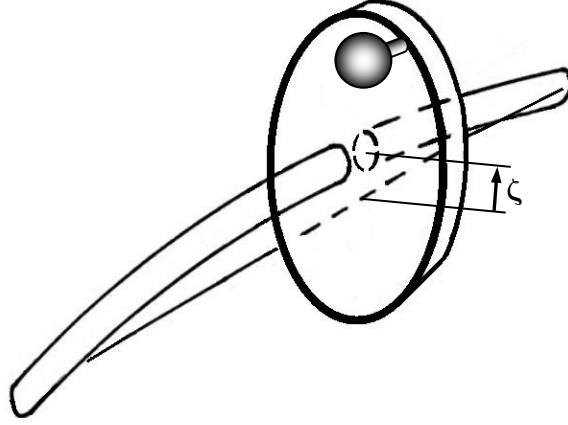


Figure 2. Whirling rotor due to imbalance and a flexible shaft.

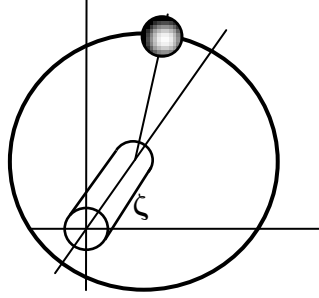


Figure 3. Axial view of rotor whirl.

The mass distribution and offsets used in Example 1 are retained, but the 30° disk tilt is omitted in this example. Thus the zeroth, first, and second mass moments are those of the untilted rotor, before the rotation $\mathbf{Q}$ applied in Example 1. These moments need not be recalculated; the first moment determines the center-of-mass offset and the rotating transverse excitation, while the second moment is used subsequently in the kinetic-energy calculation.

Let the location of the extraneous spherical mass attached to the disk be still denoted by $\mathbf{e}$ and at the instant shown be given by (60). And let the bore $\mathbf{b}$ still be given by (57). Then

$$\mathbf{e}_o = \begin{bmatrix} -0.003 \\ -0.200 \\ 0.050 \end{bmatrix} \text{ m}, \quad \mathbf{b} = \begin{bmatrix} -0.003 \\ 0 \\ 0 \end{bmatrix} \text{ m}. \quad (78)$$

The centroid $\mathbf{c}$ of the rotor relative to the point of attachment to the shaft is the net effect of the bore misalignment $\mathbf{b}$ and the offset of extraneous mass at $\mathbf{e}$:

$$\mathbf{c}_o = \frac{m_d \mathbf{b}_o + m_e \mathbf{e}_o}{m_d + m_e} = \frac{\mathbf{m}_1}{m_0}, \quad (79)$$

where $\mathbf{m}_1$ was computed in (64) for Example 1 to be

$$\mathbf{m}_1 = \begin{bmatrix} -0.0117 \\ -0.0400 \\ 0.0100 \end{bmatrix} \text{ kg m} . \quad (80)$$

Therefore,

$$\mathbf{c}_o = \begin{bmatrix} -0.003 \\ -0.01026 \\ 0.002564 \end{bmatrix} . \quad (81)$$

As time passes,

$$\mathbf{c}(t) = \mathbf{\Psi}(\Omega t) \mathbf{c}_o , \quad (82)$$

where

$$\mathbf{\Psi}(\Omega t) = \begin{bmatrix} \cos \Omega t & -\sin \Omega t & 0 \\ \sin \Omega t & \cos \Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} . \quad (83)$$

Therefore,

$$\mathbf{c}(t) = \begin{bmatrix} c_{ox} \cos \Omega t - c_{oy} \sin \Omega t \\ c_{ox} \sin \Omega t + c_{oy} \cos \Omega t \\ c_{oz} \end{bmatrix} = \begin{bmatrix} -0.0030 \cos \Omega t + 0.010256 \sin \Omega t \\ -0.0030 \sin \Omega t - 0.010256 \cos \Omega t \\ 0.002564 \end{bmatrix} . \quad (84)$$

Then the location $\mathbf{r}_C$ of the center of mass of the rotor relative to the origin fixed midway between the rigid bearings is:

$$\mathbf{r}_C = \boldsymbol{\zeta} + \mathbf{c} . \quad (85)$$

Differentiating gives

$$\dot{\mathbf{r}}_C = \dot{\boldsymbol{\zeta}} + \dot{\mathbf{c}} , \quad (86)$$

where

$$\dot{\mathbf{c}} = \mathbf{\Omega} \mathbf{c} = \Omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{c} = \Omega \begin{bmatrix} -c_{ox} \sin \Omega t - c_{oy} \cos \Omega t \\ c_{ox} \cos \Omega t - c_{oy} \sin \Omega t \\ 0 \end{bmatrix} \quad (87)$$

5.2.1 Stress and Moments

The elastic strain energy density in the flexible shaft is (Morton 2026)

$$w = \frac{1}{2} E(\eta_{11}^2 + \eta_{22}^2 + \eta_{33}^2) + \frac{1}{2} G[(\eta_{12} + \eta_{21})^2 + (\eta_{23} + \eta_{32})^2 + (\eta_{31} + \eta_{13})^2] , \quad (88)$$

where

$$\eta_{ki} = \Gamma_{kj,i} (r_j - \bar{r}_j) + \epsilon_{k,i} ,$$

and $\bar{\mathbf{r}}$ is the centroid of the shaft.

To compute the shaft strain energy, we must first compute the shaft strain. Following the notation used previously (Morton 2026), take the shaft axis as z and a point located in a cross-section at z given, for the moment, by

$$\tilde{\mathbf{R}} = \begin{bmatrix} \mathbf{r} - \bar{\mathbf{r}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix}. \quad (89)$$

For pure bending about the x or y axis, the z -directed SE(3) strain field in (10) is

$$\mathbf{\Gamma}_{,3} = \begin{bmatrix} 0 & 0 & \kappa_y \\ 0 & 0 & -\kappa_x \\ -\kappa_y & \kappa_x & 0 \end{bmatrix}. \quad (90)$$

where κ_x and κ_y are the sectional bending curvature about the x - and y -axes, respectively. This is just the two-axis version of the curvature generator used in the cantilever beam example discussed in Morton (2026, p. 50). Therefore the affine constitutive strain is

$$\boldsymbol{\eta}_3 = \mathbf{\Gamma}_{,3} \cdot \mathbf{R} + \boldsymbol{\epsilon}_3$$

or

$$\boldsymbol{\eta}_3 = \begin{bmatrix} 0 \\ 0 \\ -\kappa_y x + \kappa_x y \end{bmatrix}$$

Therefore,

$$\eta_{33} = \kappa_x y - \kappa_y x. \quad (91)$$

For pure shaft bending, the local strain energy density in (88) reduces to

$$w = \frac{1}{2} E \eta_{33}^2.$$

Therefore, the stress at any point in the shaft is

$$\sigma_{33} = \frac{\partial w}{\partial \eta_{33}} = E \eta_{33}.$$

or

$$\sigma_{33} = E(\kappa_x y - \kappa_y x). \quad (92)$$

To find the moment about x , integrate the stress moment over the cross section. The moment about x is

$$\mathbf{M} = \mathbf{\Pi}_{\text{Rot}} = \int_A \frac{\partial w}{\partial \mathbf{\Gamma}} dA, \quad (93)$$

where the derivative of (88) is (Morton 2026)

$$\frac{\partial w}{\partial \Gamma_{pq,r}} = \begin{cases} E \eta_{pp} R_q & p = r \\ G(\eta_{pr} + \eta_{rp}) R_q & p \neq r \end{cases}$$

According to (91), the only non-zero element of $\boldsymbol{\eta}$ is the lower right element so that the index $p = r$; therefore, substitute (91) into the above relation and get, when $q = 2$,

$$M_x = E \kappa_x \int y^2 dA - E \kappa_y \int x y dA, \quad (94)$$

which happens to be

$$M_x = \int y \sigma_{33} dA.$$

Therefore,

$$M_x = E(I_x \kappa_x - I_{xy} \kappa_y).$$

Similarly, when $q = 1$,

$$\begin{aligned} M_y &= - \int x \sigma_{33} dA \\ &= E(I_y \kappa_y - I_{xy} \kappa_x) \end{aligned}$$

For the circular shaft,

$$I_x = I_y = I, \quad I_{xy} = 0$$

This simplifies the moments to

$$M_x = E I \kappa_x, \quad M_y = E I \kappa_y. \quad (95)$$

For a solid circular shaft,

$$\begin{aligned} I &= \int y^2 dA = \int x^2 dA = \frac{\pi D^4}{64} \\ I &= 6.36173 \times 10^{-7} \text{ m}^4 \end{aligned} \quad (96)$$

The bearings are located at

$$z = 0, \quad z = L,$$

and the rotor is located at

$$z = \frac{L}{2}.$$

Therefore the sectional bending moment about x is

$$M_x(z) = \begin{cases} \frac{F_y}{2} z, & 0 \leq z \leq L/2 \\ \frac{F_y}{2} (L - z), & L/2 \leq z \leq L \end{cases} \quad (97)$$

Substituting this into (95) gives the curvature κ_x :

$$\kappa_x(z, t) = \begin{cases} \frac{F_y(t)}{2EI} z, & 0 \leq z \leq L/2 \\ \frac{F_y(t)(L - z)}{2EI}, & L/2 \leq z \leq L \end{cases}$$

By similar reasoning,

$$\kappa_y(z, t) = -\frac{F_x(t)}{2EI} \begin{cases} z, & 0 \leq z \leq L/2 \\ (L - z), & L/2 \leq z \leq L \end{cases}$$

The local strain (91) can be written

$$\eta_{33}(x, y, z, t) = \kappa_x(z, t) y - \kappa_y(z, t) x,$$

therefore, the general time-dependent two-axis bending solution is

$$\sigma_{33} = E[\kappa_x(z, t)y - \kappa_y(z, t)x].$$

5.2.2 Reconstructing the Shaft Displacement

The bending angle β is the integral of curvature in the z direction. For half of the shaft,

$$\begin{aligned}\beta_x(z) &= \beta_x(0) + \int_0^z \kappa_x(s) ds = \beta_x(0) + \frac{F_y}{2EI} \int_0^z s ds \\ &= \beta_x(0) + \frac{F_y z^2}{4EI}\end{aligned}\tag{98}$$

The support constrains displacement, but not cross-sectional rotation. By symmetry, the slope at the center vanishes: $\beta_x(L/2) = 0$. Therefore,

$$\beta_x(0) = -\frac{F_y L^2}{16EI}$$

Substituting this back into the relation above means that

$$\beta_x(z) = \frac{F_y}{4EI} \left(z^2 - \frac{L^2}{4} \right).$$

So unlike the cantilever beam, for which $\beta_x(0) = 0$ at the clamped end, here, the unknown integration constant is determined by symmetry at the midpoint.

To construct the shaft displacement, integrate again

$$\zeta_y(z) = \zeta_y(0) - \int_0^z \beta_x(s) ds.$$

This reconstruction uses the small-slope beam approximation $d\zeta_y/dz \approx -\beta_x$. The simply supporting bearing gives $\zeta_y(0) = 0$. Therefore,

$$\zeta_y(z) = \frac{F_y}{4EI} \int_0^z \left(\frac{L^2}{4} - s^2 \right) ds$$

or

$$\zeta_y(z) = \frac{F_y}{4EI} \left(\frac{L^2 z}{4} - \frac{z^3}{3} \right). \quad (0 \leq z \leq L/2) \tag{99}$$

At the rotor location, the magnitude of the displacement is

$$|\zeta_y(L/2)| = \frac{F_y}{4EI} \left| \frac{L^3}{24} - \frac{L^3}{8} \right| \tag{100}$$

The force in the y -direction is

$$|F_y| = \frac{48EI}{L^3} |\zeta_y|.$$

Thus, the spring constant is

$$k = \frac{48EI}{L^3} = 4.8858 \times 10^7 \text{ N/m}. \tag{101}$$

So for $m = m_d + m_e = 3.9 \text{ kg}$, the undamped natural frequency of translation is

$$\omega_N = \sqrt{\frac{k}{m}} = 3539.45 \text{ rad/s}. \quad (102)$$

At 4000 rpm, this is about 12% of the translational critical speed.

The restoring force in the shaft is

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = -\frac{48EI}{L^3} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

and

$$\mathbf{F} = -k\boldsymbol{\zeta}. \quad (103)$$

For the ideal circular shaft *with isotropic supports* considered here, the same stiffness operates in every transverse direction, and the steady synchronous response to a rotating imbalance is circular. In an actual rotor-bearing system, the effective transverse stiffnesses generally need not be equal. Replacing k by a transverse stiffness matrix $\mathbf{K}$ permits the same rotating imbalance vector to generate an elliptical orbit via an anisotropic bearing and support stiffness which causes different dynamic responses in the two transverse directions. A common source of this is anisotropy of the bearing/support system—different horizontal and vertical bearing/support stiffnesses. Fluid-film journal bearings complicate the situation further because the bearing force generally has direct and cross-coupled stiffness/damping coefficients. Differences in K_{xx} and K_{yy} , together with cross-coupled coefficients, commonly produce elliptical shaft orbits. In the simplest anisotropic case

$$\mathbf{K} = \begin{bmatrix} K_{xx} & 0 & 0 \\ 0 & K_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The elasticity paper (Morton 2026) used ζ as the displacement of a material point, whereas a was frame translation; however, they coincide here, not as a general identity, but because ζ has specifically been defined as displacement of the rotor/shaft reference point. Consequently, for the particular motion examined in this example,

$$\boldsymbol{\Theta}(t) = \boldsymbol{\Psi}(t),$$

where $\boldsymbol{\Theta}$ appeared in (2), and as noted in (4), the rotational part of the temporal generator is

$$\boldsymbol{\Omega} = \boldsymbol{\Psi}^T \dot{\boldsymbol{\Psi}}.$$

So unlike the previous example, we will now have a velocity due to the orbiting shaft, which, like (5), is

$$\mathbf{v} = \boldsymbol{\Psi}^T \dot{\boldsymbol{\zeta}} = \boldsymbol{\Psi}^T \begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \\ 0 \end{bmatrix}. \quad (104)$$

Incidentally, by differentiating (112) and (113) below, we can find that for a circular orbit,

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\Omega} \boldsymbol{\zeta}. \quad (105)$$

5.2.3 Rotor Dynamics Determines Orbit Shape

Let the rotating center-of-mass offset be of the form

$$\mathbf{c}(t) = \begin{bmatrix} c_{ox} \cos \Omega t - c_{oy} \sin \Omega t \\ c_{ox} \sin \Omega t + c_{oy} \cos \Omega t \\ c_z \end{bmatrix} \quad (106)$$

or equivalently,

$$\mathbf{c}(t) = \mathbf{\Psi}(\Omega t) \mathbf{c}_o. \quad (107)$$

Here, $\mathbf{c}_o$ is given by (81). Therefore, for constant $\mathbf{\Omega}$,

$$\ddot{\mathbf{c}}(t) = \mathbf{\Omega}^2 \mathbf{c}. \quad (108)$$

Differentiating (86) again gives

$$\ddot{\mathbf{r}}_C = \ddot{\boldsymbol{\zeta}} + \mathbf{\Omega}^2 \mathbf{c}.$$

Newton's law can then be written by equating $m \ddot{\mathbf{r}}_C$ to (103), giving

$$m(\ddot{\boldsymbol{\zeta}} + \mathbf{\Omega}^2 \mathbf{c}) = -k \boldsymbol{\zeta}$$

or

$$m \ddot{\boldsymbol{\zeta}} + k \boldsymbol{\zeta} = -m \mathbf{\Omega}^2 \mathbf{c}(t) = m \omega^2 \mathbf{c}_\perp(t), \quad (109)$$

where

$$\mathbf{c}_\perp = \begin{bmatrix} c_x \\ c_y \\ 0 \end{bmatrix}.$$

Incidentally, multiplication by the scalar ω^2 instead of $\mathbf{\Omega}^2$ would have given a spurious nonzero axial force $m \omega^2 c_z$, whereas squaring the skew-symmetric angular velocity matrix $\mathbf{\Omega}$ gives

$$\mathbf{\Omega}^2 = \begin{bmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so takes the appropriate projection which leads to the full three-dimensional forcing vector

$$-m \mathbf{\Omega}^2 \mathbf{c}(t) = m \omega^2 \begin{bmatrix} c_x(t) \\ c_y(t) \\ 0 \end{bmatrix}.$$

By substituting (83) into (109), we get

$$m \ddot{\zeta}_x + k_x \zeta_x = m \Omega^2 [c_{ox} \cos(\Omega t) - c_{oy} \sin(\Omega t)], \quad (110)$$

$$m \ddot{\zeta}_y + k_y \zeta_y = m \Omega^2 [c_{ox} \sin(\Omega t) + c_{oy} \cos(\Omega t)]. \quad (111)$$

If we set $k_y = k_x$, these equations recover the conventional undamped Jeffcott-rotor equations (Jeffcott 1919; Friswell *et al.* 2010, Eq. 6.16, pp. 234, 241). The steady synchronous particular solution is

$$\zeta_x = \frac{m\Omega^2}{k_x - m\Omega^2} [c_{ox} \cos(\Omega t) - c_{oy} \sin(\Omega t)], \quad (112)$$

$$\zeta_y = \frac{m\Omega^2}{k_y - m\Omega^2} [c_{ox} \sin(\Omega t) + c_{oy} \cos(\Omega t)], \quad (113)$$

Define $c_\perp = \sqrt{c_{ox}^2 + c_{oy}^2}$ and

$$A_x = \frac{m\Omega^2 c_\perp}{k_x - m\Omega^2}, \quad A_y = \frac{m\Omega^2 c_\perp}{k_y - m\Omega^2}$$

results in

$$\frac{\zeta_x^2}{A_x^2} + \frac{\zeta_y^2}{A_y^2} = \cos^2(\Omega t) + \sin^2(\Omega t) = 1,$$

Since the coefficients A_x and A_y need not be equal, the orbit of the shaft can be elliptical even though Ψ describes the circular rotation of the imbalance vector.

The set of conventional undamped Jeffcott-rotor equations (110) and (111) was not found by manually introducing the imbalance force $m c \Omega^2$. It results from differentiating the translation-rotation coupling term in the kinetic energy. That coupling term is governed by the first mass moment $\mathbf{m}_1$, while the ordinary translational inertia is governed by the zeroth mass moment m_0 . Thus the same mass moments that arose directly from the SE(3) kinetic energy integral generate both the rigid-rotor imbalance of Example 1 and the lateral forcing of the flexible-shaft rotor.

In this example, the rotating first moment in (33) corresponds to $m \mathbf{c}(t)$, and $m_0 = m$. Therefore, the physical linear momentum in (33) is

$$\mathbf{p} = m \dot{\boldsymbol{\zeta}} + m \dot{\mathbf{c}}.$$

Differentiating gives

$$\dot{\mathbf{p}} = m \ddot{\boldsymbol{\zeta}} + m \ddot{\mathbf{c}}. \quad (114)$$

And by (107),

$$\ddot{\mathbf{c}} = \Omega^2 \Psi(t) \mathbf{c}_o.$$

Therefore

$$\dot{\mathbf{p}} = m \ddot{\boldsymbol{\zeta}} + m \Omega^2 \Psi(t) \mathbf{c}_o. \quad (115)$$

Equating this to the shaft force in (103) gives the same governing equation as in (109). Therefore, the same first mass moment $\mathbf{m}_1$ in (33) appears first as imbalance momentum in the rigid rotor and then, after differentiation, becomes the familiar rotating imbalance force of the Jeffcott rotor. The conventional rotating imbalance force is the time derivative of the first moment contribution to the linear momentum. The rigid-rotor bearing force of Example 1 and the excitation of the flexible-shaft rotor are therefore two different effects of the same first mass moment.

5.2.4 Forces

The rotating imbalance excitation is given by the term

$$\mathbf{F}(t) = -m\Omega^2 \mathbf{c}(t) \quad (116)$$

in (115). With $\mathbf{c}$ given by (106), the imbalance force becomes

$$\mathbf{F}(t) = \begin{bmatrix} -2052.88 \cos \Omega t + 7018.39 \sin \Omega t \\ -2052.88 \sin \Omega t - 7018.39 \cos \Omega t \\ 0 \end{bmatrix} \text{ N}.$$

Its magnitude is

$$\mathbf{F}(t) = 7312.46 \text{ N}. \quad (117)$$

This force emerged from the first mass moment.

The amplitude of shaft deflection ζ is found by (112) and (113) to be

$$|\zeta| = \sqrt{\left(\frac{m\Omega^2 c_{ox}}{k_x - m\Omega^2}\right)^2 + \left(\frac{m\Omega^2 c_{oy}}{k_y - m\Omega^2}\right)^2} = \frac{(3.9)(418.88)^2}{4.8858 \times 10^7 - 3.9(418.88)^2} \sqrt{0.0030^2 + 0.01026^2}$$

$$|\zeta| = 0.00015179 \text{ m}. \quad (118)$$

Therefore the force $F^{(S)}$ transmitted through the shaft is found by substituting the above displacement and (101) into (103). This gives

$$\begin{aligned} F^{(S)} &= (4.8858 \times 10^7 \text{ N/m})(0.00015179 \text{ m}) \\ &= 7416.2 \text{ N} \end{aligned}$$

Substituting this force into (97) gives the maximum bending moment at the shaft midpoint of 927.0 N·m. Then solve for the curvature in (95), and substitute the result into (92) to obtain the maximum bending stress of 43.7 MPa.

The common force component at each bearing has magnitude

$$\frac{F^{(S)}}{2} = 3708.1 \text{ N}.$$

The force of the imbalanced 0.2-kg eccentric mass is given by

$$\mathbf{F}^{(e)}(t) = -m_e \Omega^2 \mathbf{e}_o,$$

where $\mathbf{e}_o$ was given in (78). Then

$$\mathbf{F}^{(e)} = -m_e \Omega^2 \begin{bmatrix} -0.003 \\ -0.200 \\ 0.050 \end{bmatrix} = \begin{bmatrix} -105.276 \\ -7018.385 \\ 0 \end{bmatrix} \text{ N}.$$

From (112) and (113), the shaft displacement is

$$\zeta(t) = \frac{m\Omega^2}{k - m\Omega^2} \mathbf{c}_\perp(t). \quad (119)$$

From (116), the amplitude of the imbalance force is

$$F_{\text{imb}} = m\Omega^2 c_{\perp}$$

Therefore, the magnitude of (119) can be written

$$|\zeta| = \frac{F_{\text{imb}}}{k - m\Omega^2}$$

Substituting this into (103) gives a relation between the magnitudes of the shaft restoring force and the imbalance force:

$$F^{(S)} = \frac{k}{k - m\Omega^2} F_{\text{imb}}$$

or, using (102),

$$\frac{F^{(S)}}{F_{\text{imb}}} = \frac{1}{1 - (\Omega/\omega_N)^2}.$$

So as the operating speed decreases below the shaft's natural frequency, the transmitted shaft force approaches the imbalance excitation. Conversely, the undamped force amplification factor increases without bound as the operating speed approaches the critical speed from below. Notice that in this example, $F^{(S)} = 7416.2$ N whereas the imbalance force in (117) is 7312.46 N.

5.2.5 Moments

The moment vector about the shaft attachment point is

$$\boldsymbol{\tau}^{(e)}(0) = \mathbf{e}_o \times \mathbf{F}^{(e)}(0) = \begin{bmatrix} 350.919 \\ -5.2638 \\ 0 \end{bmatrix} \text{ N} \cdot \text{m}$$

Thus

$$\tau^{(e)} = 350.959 \text{ N} \cdot \text{m}.$$

It rotates with the rotor as follows:

$$\tau_x^{(e)}(t) = 350.919 \cos \Omega t + 5.26379 \sin \Omega t.$$

$$\tau_y^{(e)}(t) = 350.919 \sin \Omega t - 5.26379 \cos \Omega t.$$

The moment excitation calculated here is not included in the two-degree-of-freedom Jeffcott shaft response model. Its effect on shaft deformation would require additional rotational degrees of freedom.

5.2.6 Shaft Strain Energy

The strain energy stored in the shaft is

$$U_S = \frac{1}{2} k \zeta^T \zeta, \quad (120)$$

where k is given by (101), and the displacement is given by (118). This results in

$$U_S = 0.5629 \text{ J}.$$

Although the shaft is treated as massless, it stores elastic energy, while the kinetic energy of

the idealized system is attributed to the rigid rotor.

5.2.7 Kinetic Energy

The complete kinetic energy T of the system, with the supporting shaft treated as massless, can be written compactly in terms of the homogeneous mass-moment matrix $\mathbf{J}$:

$$T = \frac{1}{2} \text{tr}(\xi_0 \mathbf{J} \xi_0^T), \quad (121)$$

where ξ_0 is the temporal velocity generator given by (6), and $\mathbf{J}$, given by (22) and (23), contains the zeroth, first, and second mass moments:

$$\mathbf{J} = \int \tilde{\mathbf{R}} \tilde{\mathbf{R}}^T dm = \begin{bmatrix} \mathbf{m}_2 & \mathbf{m}_1 \\ \mathbf{m}_1^T & m \end{bmatrix}.$$

Equation (121) is a generalization of (38).

The homogeneous mass-moment matrices have been used in kinetic energy formulations in robot dynamics in the past. As noted in the introduction, Paul (1981) and Fu *et al.* (1987, p. 89) expressed kinetic energy in trace form using the 4×4 pseudo-inertia matrix. Wensing *et al.* (2018, p. 64) demonstrated the equivalence of the kinetic-energy expressions using a 6×6 spatial-inertia matrix and a 4×4 pseudo-inertia matrix. The latter is equivalent to the homogeneous mass-moment matrix used here. However, the present development instead obtains the individual zeroth-, first-, and second-moment energy contributions directly from the temporal SE(3) velocity generator before assembling them into the compact form (121) and using the temporal velocity generator ξ_0 given by (6). To compute (121), first multiply $\xi_0 \mathbf{J}$:

$$\begin{aligned} \xi_0 \mathbf{J} &= \begin{bmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{m}_2 & \mathbf{m}_1 \\ \mathbf{m}_1^T & m \end{bmatrix} \\ &= \begin{bmatrix} \Omega \mathbf{m}_2 + \mathbf{v} \mathbf{m}_1^T & \Omega \mathbf{m}_1 + m \mathbf{v} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

The upper-right block reproduces the linear momentum in (33). Since

$$\xi_0^T = \begin{bmatrix} \Omega^T & 0 \\ \mathbf{v}^T & 0 \end{bmatrix},$$

the matrix on the right side of (121) then becomes

$$\xi_0 \mathbf{J} \xi_0^T = \begin{bmatrix} (\Omega \mathbf{m}_2 + \mathbf{v} \mathbf{m}_1^T) \Omega^T + (\Omega \mathbf{m}_1 + m \mathbf{v}) \mathbf{v}^T & 0 \\ 0 & 0 \end{bmatrix}$$

so that

$$\begin{aligned} T &= \frac{1}{2} \text{tr}(\Omega \mathbf{m}_2 \Omega^T) + \frac{1}{2} \text{tr}(\mathbf{v} \mathbf{m}_1^T \Omega^T) + \frac{1}{2} \text{tr}(\Omega \mathbf{m}_1 \mathbf{v}^T) + \frac{1}{2} m_0 \text{tr}(\mathbf{v} \mathbf{v}^T) \\ &= \frac{1}{2} \text{tr}(\Omega^T \Omega \mathbf{m}_2) + \mathbf{v}^T \Omega \mathbf{m}_1 + \frac{1}{2} m_0 \mathbf{v}^T \mathbf{v} \end{aligned}$$

Into this relation substitute (53) and the following

$$m_0 = m, \quad \mathbf{m}_1 = m\mathbf{c}$$

to obtain

$$T = \frac{1}{2} \text{tr}(\boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{m}_{2,C}) + \frac{1}{2} m \text{tr}(\boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{c} \mathbf{c}^T) + m \mathbf{v}^T \boldsymbol{\Omega} \mathbf{c} + \frac{1}{2} m \mathbf{v}^T \mathbf{v}.$$

And since

$$\text{tr}(\boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{c} \mathbf{c}^T) = \mathbf{c}^T \boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{c} = (\boldsymbol{\Omega} \mathbf{c})^T (\boldsymbol{\Omega} \mathbf{c}),$$

we get

$$T = \underbrace{\frac{1}{2} \text{tr}(\boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{m}_{2,C})}_{\substack{\text{classical rotation} \\ \text{about center of mass} \\ T_1}} + \underbrace{\frac{1}{2} m (\boldsymbol{\Omega} \mathbf{c})^T (\boldsymbol{\Omega} \mathbf{c})}_{\substack{\text{rotation of COM} \\ \text{about shaft} \\ T_2}} + \underbrace{m \mathbf{v}^T \boldsymbol{\Omega} \mathbf{c}}_{\substack{\text{shaft translation/} \\ \text{rotation coupling} \\ T_3}} + \underbrace{\frac{1}{2} m \mathbf{v}^T \mathbf{v}}_{\substack{\text{shaft whirl} \\ \text{about bearing} \\ \text{axis, } T_4}}. \quad (122)$$

The translational term is the far-right term:

$$T_4 = T_{\text{tr}} = T_{\text{whirl}} = \frac{1}{2} m (\dot{\zeta}_1^2 + \dot{\zeta}_2^2).$$

This rotor whirl velocity is governed by the zeroth mass moment.

The coupling term for the interaction between the translating shaft and the rotating center of mass is, according to (87) and (104),

$$T_3 = T_{\text{cpl}} = m \dot{\mathbf{c}}^T \dot{\boldsymbol{\zeta}}.$$

For the isotropic circular synchronous orbit, (105) gives

$$\begin{aligned} |T_3| &= m \omega c_{\perp} \dot{\zeta} = m c_{\perp} \omega^2 \zeta = (3.9)(0.01069)(418.88)^2(0.0001518) \\ &= 1.110 \text{ J}. \end{aligned} \quad (123)$$

The second term is the kinetic energy associated with the complete rotor's center of mass circulating about the shaft attachment point:

$$T_2 = T_{\text{ecc}} = \frac{1}{2} m \dot{\mathbf{c}}^T \dot{\mathbf{c}}. \quad (124)$$

For constant angular velocity Ω and constant center-of-mass offset $\mathbf{c}_{\perp}$, it is

$$\begin{aligned} T_{\text{ecc}} &= \frac{1}{2} m c^2 \Omega^2 \\ T_{\text{ecc}} &= \frac{1}{2} (3.9)(0.01069)^2 (418.88)^2 \\ &= 39.1 \text{ J} \end{aligned} \quad (125)$$

In the ideal constant-speed Jeffcott problem, this kinetic energy component remains constant.

The first term in the kinetic energy can be computed by using the untilted second mass moment $\mathbf{m}_2$ from (62). The complete rotor's second moment about its center of mass is then

$$\mathbf{m}_{2,C} = \mathbf{m}_2 - \frac{\mathbf{m}_1 \mathbf{m}_1^T}{m_0}.$$

Substituting values from (80) and (62) gives

$$\mathbf{m}_{2,C} = \begin{bmatrix} 0.03700 & 0 & 0 \\ 0 & 0.044590 & -0.001897 \\ 0 & -0.001897 & 0.0007519 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} T_1 &= \frac{1}{2} \omega^2 (m_{2,C,xx} + m_{2,C,yy}) \\ T_1 &= \frac{1}{2} 418.88^2 (0.03700 + 0.044590) \\ &= 7157.85 \text{ J} \end{aligned}$$

The characteristic feature of (122) is that it was not assembled by appending translational, rotational, and imbalance energies. All four terms result from the single homogeneous mass-moment matrix $\mathbf{J}$ acted upon by the temporal SE(3) generator. The zeroth mass moment produces the kinetic energy of shaft translation, the first mass moment produces the translation-rotation coupling associated with imbalance, and the second mass moment produces both rotation about the center of mass and the energy associated with rotation of the displaced center of mass about the shaft. Thus, no separate rotational term need be added after computing the center-of-mass motion, nor must the conventional rotating imbalance force be prescribed independently. Both arise from the same mass distribution and temporal SE(3) kinematics.

This result also shows why retaining the homogeneous mass-moment matrix is useful for an imperfect rotor. A change of reference point modifies the first and second mass moments simultaneously, while the form of the kinetic-energy expression itself remains unchanged.

Although (122) contains four energy terms, the center-of-mass velocity relation (86) provides an alternative interpretation. We could view the system as merely the shaft velocity and the movement of the center of mass around the shaft:

$$\dot{\mathbf{r}}_C = \dot{\boldsymbol{\zeta}} + \dot{\mathbf{c}}$$

Then, its translational kinetic energy is

$$T_C = \frac{1}{2} m (\dot{\boldsymbol{\zeta}} + \dot{\mathbf{c}})^T (\dot{\boldsymbol{\zeta}} + \dot{\mathbf{c}})$$

Expanding gives

$$T_C = \frac{1}{2} m \dot{\mathbf{c}}^T \dot{\mathbf{c}} + m \dot{\boldsymbol{\zeta}}^T \dot{\mathbf{c}} + \frac{1}{2} m \dot{\boldsymbol{\zeta}}^T \dot{\boldsymbol{\zeta}}.$$

These are terms 2, 3, and 4 of (122), expressed spatially.

The kinetic energy calculated in §5.1.2 for Example 1 includes both T_1 and T_2 of (122). Since the shaft reference point is fixed in that example, $T_3 = T_4 = 0$.

Equation (122) contains the kinetic-energy contributions of the rotor, while the elastic strain energy stored in the flexible shaft is given by (120) as $U_S = k \boldsymbol{\zeta}^T \boldsymbol{\zeta} / 2$. Thus the total mechanical energy of the ideal undamped rotor-shaft system is

$$E = T + U_S.$$

In steady circular synchronous whirl, both $\zeta^T \dot{\zeta}$ and $\dot{\zeta}^T \dot{\zeta}$ (both U and T) are constant, even though the vectors producing them rotate with time. Table 1 lists the relative size of the various energy contributions.

Term from (122)	Energy	Value Relative to U_S
T_1 : rotation about center of mass	7158 J	13,000
T_2 : eccentric center of mass rotation	39.1 J	69.4
T_3 : rotation/translation coupling	1.110 J	1.972
U_S : shaft strain energy	0.563 J	1
T_4 : shaft-whirl translation	0.00788 J	0.014

Table 1. Kinetic energy levels of the various terms in (122) and values relative to the strain energy in the shaft.

6. Conclusion

A temporal SE(3) formulation of rotor imbalance and vibration has been developed in which the zeroth, first, and second mass moments emerge from a single kinetic energy integral. These moments govern translational inertia, translation-rotation coupling, and rotational inertia, respectively. Differentiation of kinetic energy with respect to the temporal Lie algebra generator ξ_0 produces the linear and matrix angular momenta as components of a unified momentum wrench. The conventional rotational inertia matrix and parallel-axis theorem follow from the underlying mass moments.

The rigid-rotor example demonstrates that the first mass moment determines the resultant rotating imbalance force, while the second mass moment determines the associated transverse moment excitation. These forces and moments are obtained by differentiating the kinetic energy, without first reducing the angular-momentum matrix to an axial vector. Extending the analysis to a flexible shaft recovers the conventional Jeffcott-rotor equations, with the rotating imbalance force arising from the same first mass moment rather than being independently prescribed.

The homogeneous mass-moment formulation separates rotor kinetic energy into rotation about the center of mass and translation of the center of mass. The latter is further decomposed into contributions from eccentric rotation about the shaft attachment point, translation of that point during whirl about the bearing axis, and their velocity coupling (cross terms). Combined with the elastic strain energy stored in the supporting shaft, these terms provide a unified accounting of the mechanical energy of the idealized rotor-shaft system. Thus, the formulation connects the mass distribution of an imperfect rotor to its inertial excitation and elastic response while remaining consistent with established rotordynamics results. The essential result is that rotor imbalance forces, moment excitation, and the associated vibration response can be traced to the same underlying mass distribution, rather than being introduced as separate mechanical effects.

References

- Balafoutis, C. A., Patel, R. V., & Misra, P., "Efficient Modeling and Computation of Manipulator Dynamics Using Orthogonal Cartesian Tensors," *IEEE Journal on Robotics and Automation* **4** (1988) pp. 665-676.
- Featherstone, R., "The Calculation of Robot Dynamics Using Articulated-Body Inertias," *The International Journal of Robotics Research* **2** (1983) pp. 13-30.
- Featherstone, R., *Robot Dynamics Algorithms*, Kluwer Academic Publishers (1987).
- Featherstone, R., *Rigid Body Dynamics Algorithms*, Springer (2008).
- Friswell, M. I., Penny, J. E. T., Garvey, S. D., & Lees, A. W., *Dynamics of Rotating Machines*, Cambridge University Press, Cambridge (2010).
- Fu, K. S., Gonzalez, R. C., & Lee, C. S. G., *Robotics: Control, Sensing, Vision, and Intelligence*, McGraw-Hill, New York (1987).
- Genta, G., *Dynamics of Rotating Systems*, Springer (2005).
- Jeffcott, H. H., "The Lateral Vibration of Loaded Shafts in the Neighborhood of a Whirling Speed.—The Effect of Want of Balance," *Philosophical Magazine* **37** (1919) pp. 304-314.
- Haslam, A. H., Schwingshackl, C. W., & Rix, A. I. J., "A parametric study of an unbalanced Jeffcott rotor supported by a rolling-element bearing," *Nonlinear Dynamics* **99** (2020) pp. 2571-2604.
- Holm, D. D., *Geometric Mechanics Part II: Rotating, Translating and Rolling*, 2nd ed., Imperial College Press (2011).
- Laus, L. P. & Selig, J. M., "Rigid body dynamics using equimomental systems of point-masses," *Acta Mechanica* **231** (2020) pp. 221-236.
- Manakov, S. V., "Note on the integration of Euler's equations of the dynamics of an n -dimensional rigid body," *Functional Analysis and Its Applications* **10** (1976) pp. 328-329.
- Morton, T. S., "Continuous analytical elasticity solutions for arbitrary cross-sections and combined loads," *Journal of Scientific and Mathematical Research* **6** (2026) pp. 15-64.
- Murray, R. M., Li, Z., & Sastry, S. S., *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Boca Raton (1994).
- Nelson, H. D. & McVaugh, J. M., "The Dynamics of Rotor-Bearing Systems Using Finite Elements," *Journal of Engineering for Industry* **98** (1976) pp. 593-600.
- Park, F. C., Bobrow, J. E., & Ploen, S. R., "A Lie Group Formulation of Robot Dynamics," *The International Journal of Robotics Research* **14** (1995) pp. 609-618.
- Park, F. C. & Kim, M. W., "Lie theory, Riemannian geometry, and the dynamics of coupled rigid bodies," *Zeitschrift für angewandte Mathematik und Physik (ZAMP)* **51** (2000) pp. 820-834.
- Paul, R. P., *Robot Manipulators: Mathematics, Programming, and Control*, MIT Press, Cambridge, MA (1981).
- Simmonds, J. G., *A Brief on Tensor Analysis*, Springer-Verlag, New York (1982).
- Uicker, Jr., J. J., *On the Dynamic Analysis of Spatial Linkages using 4×4 Matrices*, Dissertation, Northwestern University (1965).
- Uicker, Jr., J. J., "Dynamic Force Analysis of Spatial Linkages," *Journal of Applied Mechanics* **34** (1967) pp. 418-424.
- Wensing, P. M., Kim, S., & Slotine, J.-J. E., "Linear Matrix Inequalities for Physically Consistent Inertial Parameter Identification: A Statistical Perspective on the Mass Distribution," *IEEE Robotics and Automation Letters* **3** (2018) pp. 60-67.
- Wittenburg, J., *Dynamics of Systems of Rigid Bodies*, B. G. Teubner, Stuttgart (1977).
- Ziese, C. Nitzschke, S. & Woschke, E., "Run up simulation of a full-floating ring supported Jeffcott-rotor considering two-phase flow cavitation," *Archive of Applied Mechanics* **91** (2021) pp. 777-790.