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Products of generalized symmetric q-numbers

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The product of two symmetric q-numbers is known to admit an exact expansion in terms of ordinary symmetric q-numbers. This paper extends that result to products of an arbitrary number of generalized symmetric q-numbers. The resulting identities express each product as a finite sum of ordinary symmetric q-numbers whose indices are generated by centered arithmetic progressions associated with the individual factors. The formula provides a unified representation of products of factors involving different base q and demonstrates that products of generalized symmetric q-numbers remain within the algebra generated by ordinary symmetric q-numbers.

Conventional q-numbers arise naturally throughout the theory of q-series, representation theory, combinatorics, and quantum algebra. Much of the existing literature concerns identities involving individual q-numbers or finite sums of q-numbers. Comparatively less attention has been given to explicit product expansions involving generalized symmetric q-numbers with differing q bases. Most studies of q-analogues use the conventional quantum number, defined as (see, e.g., Fraenkel 1955):

$$[n]_q = \sum_{i=0}^{n-1} q^i \,,$$

so that

$$n \to [n]_q = \frac{1-q^n}{1-q} \,. \qquad \text{(first quantization)}$$

In the present work, the symmetric, or second, quantization

$$n \to \langle n \rangle_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad \text{(second quantization)} \tag{1}$$

will be used. Quantum integers of the form

$$\langle n \rangle_q \qquad\qquad n \in \mathbb{Z} \tag{2}$$

will be called *simple*, while those of the form

$$\langle n \rangle_{q^a} \qquad\qquad n \in \mathbb{Z}\,,\ a \in \mathbb{Z}_{\geq 1}\,, \tag{3}$$

will be called *generalized*.

Multiplicative properties of quantum integers have also been studied through functional equations of the form $f_{mn}(q) = f_m(q) f_n(q^m)$, beginning with Nathanson (2003) who examined what multiplication laws and polynomial sequences satisfy this quantum multiplication equation. Nathanson (2003) found the following rule for products of two q-numbers of the first kind:

$$[mn]_q \;=\; [m]_q[n]_{q^m} \;=\; [m]_{q^n}[n]_q \qquad\qquad \forall m,n\,.$$

This relation is valid for both conventional and symmetric q-number products. Nathanson (2004) then examined the limiting formal power series that arise from those sequences. The present work considers a different aspect of multiplication: instead of classifying families satisfying a multiplicative functional equation, it proves an explicit finite expansion of products. Another distinction is that in the present work, generalized symmetric q-numbers (of the second kind) are used rather than conventional q-numbers. So it is a symmetric and centered-index analogue of the product-expansion problem.

A previous paper established an exact expansion for the product of two generalized symmetric q-numbers (Morton 2009). That result suggested the existence of a broader multiplicative structure. The present paper confirms this by deriving explicit expansion formulas for products of three generalized symmetric q-numbers and, more generally, for products containing an arbitrary number of factors. The resulting expansions express each product as a finite sum of ordinary symmetric q-numbers. The generated indices are determined by finite centered arithmetic progressions associated with the individual factors. This formulation unifies products involving different base q into a single expansion formula.

***Theorem* 1**.

For any $n_k, m \in \mathbb{Z}$, $a_k, q \in \mathbb{R}$, $i_k \in \mathbb{Z}$,

$$\langle n_1\rangle_{q^{a_1}}\langle n_2\rangle_{q^{a_2}}\cdots\langle n_m\rangle_{q^{a_m}} \;=\; \sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\cdots\sum_{i_m=0}^{n_m-1}\left\langle 1+\sum_{k=1}^{m} a_k(2i_k-(n_k-1))\right\rangle_q\,. \tag{4}$$

For the case of a product of 3 q-numbers, with $m,n,p \in \mathbb{Z}$, $a,b,c,q \in \mathbb{R}$, we have

$$\langle m\rangle_{q^a}\langle n\rangle_{q^b}\langle p\rangle_{q^c} \;=\; \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\sum_{k=0}^{p-1}\Big\langle 1+a[2i-(m{-}1)]+b[2j-(n{-}1)]+c[2k-(p{-}1)]\Big\rangle_q \tag{5}$$

Proof. Set $\alpha = 1-a(m{-}1)-b(n{-}1)-c(p{-}1)$. Then the right side of (5) becomes

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\sum_{k=0}^{p-1}\left[\frac{q^{\alpha+2ai+2bj+2ck}}{q-q^{-1}}-\frac{q^{-(\alpha+2ai+2bj+2ck)}}{q-q^{-1}}\right].$$

Now set $z_k = q^k - q^{-k}$. The above expression is then:

$$\frac{1}{z_1}\left[q^{\alpha}\sum_{i=0}^{m-1}q^{2ai}\sum_{j=0}^{n-1}q^{2bj}\sum_{k=0}^{p-1}q^{2ck}-q^{-\alpha}\sum_{i=0}^{m-1}q^{-2ai}\sum_{j=0}^{n-1}q^{-2bj}\sum_{k=0}^{p-1}q^{-2ck}\right].$$

Each of the summations above can be replaced by the first quantization as follows:

$$\frac{1}{z_1}\Big[\,q^{\alpha}[m]_{q^{2a}}[n]_{q^{2b}}[p]_{q^{2c}}-q^{-\alpha}[m]_{q^{-2a}}[n]_{q^{-2b}}[p]_{q^{-2c}}\,\Big]$$

$$=\frac{1}{z_1}\left[q^{\alpha}\left(\frac{1-q^{2am}}{1-q^{2a}}\right)\left(\frac{1-q^{2bn}}{1-q^{2b}}\right)\left(\frac{1-q^{2cp}}{1-q^{2c}}\right)-q^{-\alpha}\left(\frac{1-q^{-2am}}{1-q^{-2a}}\right)\left(\frac{1-q^{-2bn}}{1-q^{-2b}}\right)\left(\frac{1-q^{-2cp}}{1-q^{-2c}}\right)\right]$$

$$= \frac{q^{\alpha}(q^{am+bn+cp})}{z_1} \frac{(q^{-am}-q^{am})}{q^a(q^{-a}-q^a)} \frac{(q^{-bn}-q^{bn})}{q^b(q^{-b}-q^b)} \frac{(q^{-cp}-q^{cp})}{q^c(q^{-c}-q^c)}$$

$$- \frac{q^{-\alpha}(q^{-am-bn-cp})}{z_1} \frac{(q^{am}-q^{-am})}{q^{-a}(q^a-q^{-a})} \frac{(q^{bn}-q^{-bn})}{q^{-b}(q^b-q^{-b})} \frac{(q^{cp}-q^{-cp})}{q^{-c}(q^c-q^{-c})}$$

$$= \frac{(q^{am}-q^{-am})}{(q^a-q^{-a})} \frac{(q^{bn}-q^{-bn})}{(q^b-q^{-b})} \frac{(q^{cp}-q^{-cp})}{(q^c-q^{-c})}$$

$$= \langle m \rangle_{q^a} \langle n \rangle_{q^b} \langle p \rangle_{q^c}$$

which is the left side of (5). ■

Remark. Formula (5) is true when $m,n,p,a,b,c \in \mathbb{Z}$; however, a, b, and c need not be integers.

Example:

Let $m=3$, $n=2$, $p=4$, and $a=b=c=1$. Then (5) gives:

$$\langle 3 \rangle_q \langle 2 \rangle_q \langle 4 \rangle_q = 1 + 2\langle 3 \rangle_q + 2\langle 5 \rangle_q + \langle 7 \rangle_q$$

This works for all values of q, but if we let, say $q=0.2$, then the above relation becomes:

$$(26.04)(5.2)(130.208) = 1 + 2(26.04) + 2(651.0416) + (16,276.04)$$

$$17,631.205 \approx 17,631.203$$

The difference is, of course, round-off error.

Conclusion

The product identities presented here exhibit a rich multiplicative structure. The general product formula reveals that the indices occurring in the expansion are generated from symmetric centered arithmetic progressions of the form $2i-(n-1)$ whose linear combinations determine the indices of the resulting symmetric q-numbers. Letting $u_k = 2i_k - (n_k - 1)$ allows us to write (4) as

$$\prod_{k=1}^{m} \langle n_k \rangle_{q^{a_k}} = \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \cdots \sum_{i_m=0}^{n_m-1} \langle 1 + \sum a_k u_k \rangle_q$$

So the indices appearing in the product expansion are generated by linear combinations of centered arithmetic progressions associated with each factor.

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