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Alternating sums and half-integer structures in finite symmetric q-analogues

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A symmetric q-analogue is found for the classical finite alternating sum of consecutive numbers. Alternating sums of consecutive symmetric q-numbers exhibit parity-dependent closed forms that differ according to whether the truncation contains an even or odd number of terms. Even-length sums reduce to product expressions, whereas odd-length sums contain an additional residual term arising from the unpaired endpoint. These residual terms reveal a half-integer shift in the natural q-number index lattice associated with the alternating pairing structure and a new relevant q-number.

1. Introduction

Alternating sums of integer sequences are closely related to finite difference calculus and summation by parts. Sums of classical alternating series such as

$$\sum_{i=1}^{n} (-1)^i i \tag{1}$$

exhibit strong cancellation of interior terms due to sign oscillation. However, the symmetric q-analogue of this alternating sum exhibits additional dependence on the endpoints of the summation interval and surprising structural behavior. The symmetric q-number $\langle x\rangle_q$, sometimes called the 2nd quantization of x, is defined as

$$\langle x\rangle_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}, \qquad x \in \mathbb{Z}\,,\ q \neq 0, \pm 1\,. \tag{2}$$

In this paper, the symmetric q-analogue of (1) is presented.

A q-analogue was previously found for this series in terms of quantum integers when the series has an even number of terms and an arbitrary starting point (Kupershmidt and Morton 2011). Now it will be shown that when the alternating sum has an odd number of terms, the q-analog will expose a half-shifted index lattice due to the single unpaired residual term at the truncation boundary. This is because alternating summation separates the integer index lattice into even and odd sectors. Pairing adjacent terms produces a midpoint lattice, shifted by one-half relative to the original integer lattice.

***Theorem* 3.** For any $n \in \mathbb{Z}_{\geq 0}$,

$$\sum_{i=0}^{2n+1} (-1)^i \langle i\rangle_q = \frac{-2}{(\langle 2\rangle_{q^{1/2}})^2} - \left\langle n+\frac{1}{2}\right\rangle_q \left(\left\langle n+\frac{3}{2}\right\rangle_q - \left\langle n+\frac{1}{2}\right\rangle_q\right), \tag{4}$$

$$\sum_{i=0}^{2n} (-1)^i \langle i\rangle_q = \langle n\rangle_q (\langle n+1\rangle_q - \langle n\rangle_q)\,. \tag{5}$$

The two cases can be combined into one, with $x \in \left\{0,\ \frac{1}{2},\ 1,\ \frac{3}{2}, \ldots\right\}$, using the formula

$$\sum_{i=0}^{2x}(-1)^i\langle i\rangle_q \;=\; \frac{(-1)^{2x}-1}{(\langle 2\rangle_{\sqrt{q}})^2} \;+\; \langle x\rangle_q\Big[\langle x+1\rangle_q - \langle x\rangle_q\Big](-1)^{2x}. \tag{6}$$

Proof. Use induction on n. For the odd case, set

$$x = n + \frac{1}{2}$$

so that $2x = 2n+1$ and $(-1)^{2x} = -1$. Then, (6) becomes

$$\sum_{i=0}^{2n+1}(-1)^i\langle i\rangle_q \;=\; \frac{-2}{(\langle 2\rangle_{q^{1/2}})^2} \;-\; \left\langle n+\tfrac{1}{2}\right\rangle_q\left(\left\langle n+\tfrac{3}{2}\right\rangle_q - \left\langle n+\tfrac{1}{2}\right\rangle_q\right). \tag{7}$$

Define the left side as

$$S_{2n+1} \equiv \sum_{i=0}^{2n+1}(-1)^i\langle i\rangle_q \tag{8}$$

and the right side as

$$F(x) \;=\; \frac{-2}{(\langle 2\rangle_{\sqrt{q}})^2} \;-\; \langle x\rangle_q\Big[\langle x+1\rangle_q - \langle x\rangle_q\Big] \tag{9}$$

For $n = 0$, (6) becomes:

$$-1 \;\overset{?}{=}\; \frac{-2}{(\langle 2\rangle_{q^{1/2}})^2} - \left\langle \tfrac{1}{2}\right\rangle_q\left(\left\langle \tfrac{3}{2}\right\rangle_q - \left\langle \tfrac{1}{2}\right\rangle_q\right).$$

Multiplying both sides by $-(\langle 2\rangle_{q^{1/2}})^2$ and expanding gives

$$(\langle 2\rangle_{q^{1/2}})^2 \;\overset{?}{=}\; 2 + \frac{q^{1/2}-q^{-1/2}}{(q-q^{-1})^2}(q^{1/2}-q^{-1/2})^2(q^{3/2}-q^{-3/2}-q^{1/2}+q^{-1/2}),$$

or, with $Q = q^{1/2}$,

$$Q^2 - Q^{-2} \;\overset{?}{=}\; \frac{(Q-Q^{-1})}{(Q^2-Q^{-2})}(Q^3-Q^{-3}-Q+Q^{-1}),$$

or

$$Q^2 - Q^{-2} \;\overset{?}{=}\; \frac{1}{(Q^1-Q^{-1})}[Q(Q^2-1) - Q^{-1}(-1+Q^{-2})],$$

or

$$(Q^2-Q^{-2})(Q^1-Q^{-1}) \;\overset{?}{=}\; [QQ(Q-Q^{-1}) - Q^{-1}Q^{-1}(Q-Q^{-1})],$$

or

$$(Q^2-Q^{-2})(Q^1-Q^{-1}) \;\overset{?}{=}\; (Q^2-Q^{-2})(Q^1-Q^{-1}),$$

which is true.

For the inductive step, let $K = 2x$, so that $x \to x+1$ means $K \to K+2$ because then the parity does not change, and the constant terms cancel. So the summation up to K is

$$S_K = \sum_{i=0}^{K}(-1)^i\langle i\rangle_q ,$$

and proceeding to $K+2$ gives us two more terms:

$$S_{K+2} = \sum_{i=0}^{K+2}(-1)^i\langle i\rangle_q = \sum_{i=0}^{K}(-1)^i\langle i\rangle_q + (-1)^{K+1}\langle K+1\rangle_q + (-1)^{K+2}\langle K+2\rangle_q .$$

So

$$S_{K+2} - S_K = (-1)^{K+1}\langle K+1\rangle_q + (-1)^{K+2}\langle K+2\rangle_q .$$

Since $(-1)^{K+2} = -(-1)^{K+1}$, this is

$$S_{K+2} - S_K = (-1)^K(\langle K+2\rangle_q - \langle K+1\rangle_q) . \tag{10}$$

On the right side of (7), which is (9), the constant terms change sign and so cancel out. For the remaining terms, use the fact that $(-1)^{K+2} = (-1)^K$ to obtain

$$\begin{aligned} F_{K+2} - F_K &= (-1)^K\left\{\langle x+1\rangle_q\Big[\langle x+2\rangle_q - \langle x+1\rangle_q\Big] - \langle x\rangle_q\Big[\langle x+1\rangle_q - \langle x\rangle_q\Big]\right\} \\ &= (-1)^K\left\{\langle x+1\rangle_q\langle x+2\rangle_q - \Big[\langle x+1\rangle_q\Big]^2 - \langle x\rangle_q\langle x+1\rangle_q + \Big[\langle x\rangle_q\Big]^2\right\} \\ F_{K+2} - F_K &= (-1)^K\left\{\langle x+1\rangle_q\Big[\langle x+2\rangle_q - \langle x\rangle_q\Big] - \Big[\langle x+1\rangle_q\Big]^2 + \Big[\langle x\rangle_q\Big]^2\right\} \end{aligned} \tag{11}$$

Take one of the factors on the right, and expand it:

$$\begin{aligned} \langle x+2\rangle_q - \langle x\rangle_q &= \frac{q^{x+2} - q^{-x-2} - q^x + q^{-x}}{q - q^{-1}} \\ &= \frac{q^{x+1}(q-q^{-1}) + q^{-x}(1-q^{-2})}{q-q^{-1}} = \frac{q^{x+1}(q-q^{-1}) + q^{-x}q^{-1}(q-q^{-1})}{q-q^{-1}} \\ &= q^{x+1} + q^{-x-1} \end{aligned}$$

This gives the relation

$$\langle x+2\rangle_q - \langle x\rangle_q = q^{x+1} + q^{-(x+1)} . \tag{12}$$

This identity is interesting because it motivates the definition of a new q-number:

$$\rangle x\langle_q \equiv \frac{q^x + q^{-x}}{q + q^{-1}} . \tag{13}$$

Whereas the definition in (2) can be written, with $q = e^\lambda$, as

$$\langle n\rangle_q = \frac{\sinh(n\lambda)}{\sinh(\lambda)} ,$$

the new definition in (13) can be written as

$$\rangle n\langle_q = \frac{\cosh(n\lambda)}{\cosh(\lambda)} .$$

Note that

$$\langle 2\rangle_q \;=\; q+q^{-1},$$

so that $q^x + q^{-x} \;=\; \langle 2\rangle_q \;\rangle x\langle_q$. Now we can write (12) as

$$\langle x+2\rangle_q - \langle x\rangle_q \;=\; \langle 2\rangle_q \;\; \rangle x+1\langle_q$$

or

$$\rangle x+1\langle_q \;=\; \frac{\langle x+2\rangle_q - \langle x\rangle_q}{\langle 2\rangle_q}. \tag{14}$$

So the cosh-type q-number $\rangle x+1\langle_q$ is the analogue of "halfway" between the two-step q-interval

Now take the remaining terms on the right side of (11), and treat them as a difference of two squares:

$$\left[\langle x+1\rangle_q\right]^2 \;-\; \left[\langle x\rangle_q\right]^2 \;=\; \left[\langle x+1\rangle_q - \langle x\rangle_q\right]\left[\langle x+1\rangle_q + \langle x\rangle_q\right] \tag{15}$$

Expand one of the factors on the right:

$$\begin{aligned}
\langle x+1\rangle_q - \langle x\rangle_q \;&=\; \frac{q^{x+1} - q^{-(x+1)} - q^x + q^{-x}}{q-q^{-1}} \\
&=\; \frac{q^x(q-1) + q^{-x}(1-q^{-1})}{q-1+1-q^{-1}} \;=\; \frac{q^x(q-1)+q^{-x}q^{-1}(q-1)}{(q-1)(1+q^{-1})} \\
&=\; \frac{q}{q}\frac{(q^x + q^{-(x+1)})}{(1+q^{-1})} \;=\; \frac{(q^{x+1}+q^{-x})}{q^{1/2}(q^{1/2}+q^{-1/2})}
\end{aligned}$$

$$\langle x+1\rangle_q - \langle x\rangle_q \;=\; \frac{q^{x+1/2} + q^{-(x+1/2)}}{q^{1/2}+q^{-1/2}} \tag{16}$$

The right side is again interesting in light of the new q-number in (13). That new q-number allows (16) to be written as

$$\langle x+1\rangle_q - \langle x\rangle_q \;=\; \Big\rangle x+\tfrac{1}{2}\Big\langle_{q^{1/2}}$$

So the difference of two sinh-type q-numbers becomes a midpoint cosh-type q-number.

Now expand the other factor on the right side of (15):

$$\begin{aligned}
\langle x+1\rangle_q + \langle x\rangle_q \;&=\; \frac{q^{x+1} - q^{-(x+1)} + q^x - q^{-x}}{q-q^{-1}} \;=\; \frac{q^x q - q^{-(x+1)} + q^x - q^{-x}}{(q-1)(1+q^{-1})} \\
&=\; \frac{q^x(q+1) - q^{-x} - q^{-(x+1)}}{(q-1)q^{-1}(q+1)} \;=\; \frac{(q+1)q^x - (qq^{-(x+1)} + q^{-(x+1)})}{(q-1)q^{-1}(q+1)} \\
&=\; \frac{(q+1)q^x - (q+1)q^{-(x+1)}}{(q+1)q^{-1/2}(q^{1/2}-q^{-1/2})} \;=\; \frac{q^{1/2}(q^x - q^{-(x+1)})}{(q^{1/2}-q^{-1/2})}
\end{aligned}$$

Therefore, we have the identity:

$$\langle x+1\rangle_q + \langle x\rangle_q \;=\; \frac{q^{x+1/2} - q^{-(x+1/2)}}{q^{1/2}-q^{-1/2}}. \tag{17}$$

If we substitute (16) and (17) into the right side of (15), we get

$$\Big[\langle x+1\rangle_q\Big]^2 \;-\; \Big[\langle x\rangle_q\Big]^2 \;=\; \frac{(q^{x+1/2}+q^{-(x+1/2)})(q^{x+1/2}-q^{-(x+1/2)})}{(q^{1/2}+q^{-1/2})(q^{1/2}-q^{-1/2})}\,.$$

This can be simplified by expressing both the numerator and the denominator on the right as the difference of two squares:

$$\Big[\langle x+1\rangle_q\Big]^2 \;-\; \Big[\langle x\rangle_q\Big]^2 \;=\; \frac{q^{2x+1}-q^{-(2x+1)}}{q-q^{-1}}\,.$$

Therefore,

$$\Big[\langle x+1\rangle_q\Big]^2 \;-\; \Big[\langle x\rangle_q\Big]^2 \;=\; \langle 2x+1\rangle_q\,. \tag{18}$$

Now substitute (12) and (18) into (11)

$$\begin{aligned}
F_{K+2}-F_K \;&=\; (-1)^K\Big[\langle x+1\rangle_q(q^{x+1}+q^{-(x+1)}) \;-\; \langle 2x+1\rangle_q\Big] \\
&=\; (-1)^K\left[\frac{q^{x+1}-q^{-(x+1)}}{q-q^{-1}}(q^{x+1}+q^{-(x+1)}) \;-\; \langle 2x+1\rangle_q\right] \\
&=\; (-1)^K\left[\frac{q^{2(x+1)}-q^{-2(x+1)}}{q-q^{-1}} \;-\; \langle 2x+1\rangle_q\right] \\
&=\; (-1)^K\Big[\langle 2x+2\rangle_q \;-\; \langle 2x+1\rangle_q\Big]
\end{aligned}$$

Since $K=2x$, this is

$$F_{K+2}-F_K \;=\; (-1)^K\Big[\langle K+2\rangle_q \;-\; \langle K+1\rangle_q\Big],$$

which agrees with (10).

Examples

Consider the case when $n=2$. If $q=1$, (4) returns

$$\begin{aligned}
-\langle 1\rangle_q+\langle 2\rangle_q-\langle 3\rangle_q+\langle 4\rangle_q-\langle 5\rangle_q \;&=\; -\tfrac{1}{2} \;-\; \left\langle \tfrac{5}{2}\right\rangle_q\left(\left\langle \tfrac{7}{2}\right\rangle_q-\left\langle \tfrac{5}{2}\right\rangle_q\right) \\
-1 \;+\; 2 \;-\; 3 \;+\; 4 \;-\; 5 \;&=\; -\tfrac{1}{2} \;-\; \tfrac{5}{2}\left(\tfrac{7}{2} \;-\; \tfrac{5}{2}\right) \\
-3 \;&=\; -\,3
\end{aligned}$$

However, when $q=4$, (4) gives

$$\begin{aligned}
-\langle 1\rangle_q \;+\; \langle 2\rangle_q \;-\; \langle 3\rangle_q \;+\; \langle 4\rangle_q \;-\; \langle 5\rangle_q \;&=\; -\,(\langle 2\rangle_{q^{1/2}})^2 \;-\; \left\langle \tfrac{5}{2}\right\rangle_q\left(\left\langle \tfrac{7}{2}\right\rangle_q \;-\; \left\langle \tfrac{5}{2}\right\rangle_q\right) \\
-1+4.25-17.0625+68.2656-273.0664 \;&=\; -\,0.32 \;-\; 8.525(34.1313-8.525) \\
-218.613 \;&=\; -\,218.614
\end{aligned}$$

References

Kupershmidt, B. A. & Morton, T. S., “An alternating sum of quantum integers,” *Journal of Scientific and Mathematical Research* **5**, (2011) pp. 18-19.