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Polynomial curves on the Pythagorean quadric and their *q*-analogues

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This paper presents a one-parameter embedding of integers into the quadratic hypersurface $x^2 + y^2 + z^2 = w^2$, given by $(n,\ n{+}1,\ n(n{+}1),\ n^2{+}n{+}1)$. This embedding defines an exact algebraic curve lying entirely within the integer solution set of the equation. The identity admits a consistent symmetric q-analogue that preserves a nonlinear recurrence structure equivalent to a hyperbolic representation. The resulting framework connects polynomial Diophantine embeddings with q-analog recurrence relations of Chebyshev type.

1. Introduction

The classical Diophantine equation

$$x^2 + y^2 + z^2 = w^2 \tag{1}$$

defines a quadric hypersurface in four variables with a rich set of integer solutions, such as

$$1^2 + 2^2 + 2^2 = 3^2\,,$$

$$2^2 + 3^2 + 6^2 = 7^2\,,$$

$$3^2 + 4^2 + 12^2 = 13^2\,,$$

$$4^2 + 5^2 + 20^2 = 21^2\,.$$

This paper constructs a specific polynomial map from integers into hypersurface (1). The image is not a general parameterization of all solutions, but rather a rigid one-parameter algebraic curve entirely contained in the solution set.

2. Polynomial Embedding

Define the map

$$n \ \ \rightarrow \ \ (x,y,z,w) \,=\, (n,\ n+1,\ n(n+1),\ n^2+n+1)\,, \qquad n \in \mathbb{Z} \tag{2}$$

***Theorem* 1.** For all integers n, define

$$x = n\,, \quad y = n+1\,, \quad z = n(n+1)\,, \quad w = n^2+n+1\,. \tag{3}$$

Then

$$x^2 + y^2 + z^2 = w^2\,. \tag{4}$$

Proof

By substituting (3) into (1) and expanding the left side, we have

$$n^2+(n+1)^2+[n(n+1)]^2 \;=\; n^2+(n^2+2n+1)+(n^4+2n^3+n^2)\,.$$

Collecting terms:

$$=\; n^4+2n^3+3n^2+2n+1\,.$$

Meanwhile, the right side of (4) is

$$(n^2+n+1)^2 \;=\; n^4+2n^3+3n^2+2n+1\,.$$

Thus both sides agree. ■

The map in (2) defines a polynomial embedding of the integers into the quadratic hypersurface (1)

$$n^2+(n+1)^2+(n^2+n)^2=(n^2+n+1)^2\,,\qquad n\in\mathbb{Z}\,,\tag{5}$$

with image entirely contained in the integer solution set.

Affine Generalization

The construction extends to a one-parameter family. Let a be an integer parameter. Then expressions of the form

$$n^2+(n+a)^2+\left[n(n+a)+\frac{a^2-1}{2}\right]^2=\left[n(n+a)+\frac{a^2+1}{2}\right]^2.\tag{6}$$

define a generalized embedding family of the same quadradic structure. Note that when $a=1$, we recover (5). The relation in (6) produces a family of structured solutions rather than a single curve. Next, we introduce a q-analogue of this integer embedding.

3. *Q*-Analogue

To study the q-analogue of this structure, introduce symmetric q-numbers:

$$\langle n\rangle_q=\frac{q^n-q^{-n}}{q-q^{-1}}\,,\qquad q\in\mathbb{R}\setminus\{0,\pm 1\}\,,\tag{7}$$

also, called the 2nd quantization of n; and consider the induced q-analogue of the polynomial identity under the mapping $n\mapsto\langle n\rangle_q$.

Equivalently, writing $q=e^{\lambda}$ gives

$$\langle n\rangle_q=\frac{\sinh(n\lambda)}{\sinh(\lambda)}\,.$$

This provides a hyperbolic representation of the integer parameter.

When $q\neq 1$, most attempted analogues fail. For example, when $q=2.5$, we see that

$$\langle 2\rangle_q+\langle 5\rangle_q\overset{?}{=}\langle 3\rangle_q+\langle 4\rangle_q$$
$$49.398...\overset{?}{=}137.744...$$

So it is the startling intermittent appearance of *valid* identities with q-numbers that is most interesting.

Theorem 2. (q-Analogue of Theorem 1)

The generalized polynomial embedding of Theorem 1 admits a consistent symmetric q-analogue. Under this analogue, algebraic relations among the components of the embedding are preserved in a nonlinear form equivalent to hyperbolic addition rules. The embedding admits a symmetric q-lift preserving nonlinear closure under hyperbolic representation. That is, for any $n, a \in \mathbb{Z}$,

$$\langle n^2 + (n+a)^2 \rangle_q + \left(\left\langle n(n+a) + \frac{a^2-1}{2} \right\rangle_q \right)^2 = \left(\left\langle n(n+a) + \frac{a^2+1}{2} \right\rangle_q \right)^2 . \tag{8}$$

Proof. Use the handy formula, easily verifiable:

$$(\langle z+1 \rangle_q)^2 - (\langle z \rangle_q)^2 = \langle 2z+1 \rangle_q , \qquad z \in \mathbb{R} . \tag{9}$$

Then, the last two terms in (8) comprise:

$$\begin{aligned} \left(\left\langle n(n+a) + \frac{a^2+1}{2} \right\rangle_q \right)^2 - \left(\left\langle n(n+a) + \frac{a^2-1}{2} \right\rangle_q \right)^2 &= \left\langle 2\left[n(n+a) + \frac{a^2-1}{2} \right] + 1 \right\rangle_q \\ &= \langle 2n(n+a) + a^2 \rangle_q = \langle n^2 + (n+a)^2 \rangle_q , \end{aligned}$$

as required. ■

Example 1

As an example, write (8) with $a = 1$ as

$$\langle n^2 + (n+1)^2 \rangle_q + (\langle n(n+1) \rangle_q)^2 = (\langle n(n+1)+1 \rangle_q)^2 .$$

This is the q-analogue of (5). Now let $n = 5$. This returns the following valid identity:

$$\langle 5^2 + 6^2 \rangle_q + (\langle 5 \cdot 6 \rangle_q)^2 = (\langle 5 \cdot 6 + 1 \rangle_q)^2 ,$$

or

$$\langle 5^2 + 6^2 \rangle_q + (\langle 30 \rangle_q)^2 = (\langle 31 \rangle_q)^2 .$$

Example 2

When $n = 0$, (8) generates the table:

$$\begin{aligned} \langle 2^2 \rangle_q + \left(\left\langle \frac{3}{2} \right\rangle_q \right)^2 &= \left(\left\langle \frac{5}{2} \right\rangle_q \right)^2 , \\ \langle 3^2 \rangle_q + (\langle 4 \rangle_q)^2 &= (\langle 5 \rangle_q)^2 , \\ \langle 4^2 \rangle_q + \left(\left\langle \frac{15}{2} \right\rangle_q \right)^2 &= \left(\left\langle \frac{17}{2} \right\rangle_q \right)^2 , \\ \langle 5^2 \rangle_q + (\langle 12 \rangle_q)^2 &= (\langle 13 \rangle_q)^2 , \\ \langle 6^2 \rangle_q + \left(\left\langle \frac{35}{2} \right\rangle_q \right)^2 &= \left(\left\langle \frac{37}{2} \right\rangle_q \right)^2 \end{aligned}$$

Example 3

If we let $n = a$, (8) generates the following table:

$$\langle 1^2 + 2^2\rangle_q \ + \ (\langle 2\rangle_q)^2 \ = \ (\langle 2+1\rangle_q)^2$$

$$\langle 2^2 + 4^2\rangle_q \ + \ \left(\left\langle 8 + \frac{3}{2}\right\rangle_q\right)^2 \ = \ \left(\left\langle 8 + \frac{5}{2}\right\rangle_q\right)^2$$

$$\langle 3^2 + 6^2\rangle_q \ + \ (\langle 18+4\rangle_q)^2 \ = \ (\langle 18+5\rangle_q)^2$$

$$\langle 4^2 + 8^2\rangle_q \ + \ \left(\left\langle 32 + \frac{15}{2}\right\rangle_q\right)^2 \ = \ \left(\left\langle 32 + \frac{17}{2}\right\rangle_q\right)^2$$

$$\langle 5^2 + 10^2\rangle_q \ + \ (\langle 50+12\rangle_q)^2 \ = \ (\langle 50+13\rangle_q)^2$$

$$\vdots$$

Example 4

If $a = 3$, (8) generates the table:

$$\langle 1^2 + 4^2\rangle_q \ + \ (\langle 4+4\rangle_q)^2 \ = \ (\langle 4+5\rangle_q)^2$$

$$\langle 2^2 + 5^2\rangle_q \ + \ (\langle 10+4\rangle_q)^2 \ = \ (\langle 10+5\rangle_q)^2$$

$$\langle 3^2 + 6^2\rangle_q \ + \ (\langle 18+4\rangle_q)^2 \ = \ (\langle 18+5\rangle_q)^2$$

$$[4^2 + 7^2\rangle_q \ + \ ([28+4\rangle_q)^2 \ = \ ([28+5\rangle_q)^2$$

$$\vdots$$

Remark.

The resulting structure exhibits nonlinear recurrence behavior consistent with Chebyshev-type relations when expressed in hyperbolic variables. This suggests the existence of a structural correspondence between polynomial embeddings of integer solutions to quadratic hypersurfaces and q-analogue hyperbolic recurrence systems.