

Decompositions of Numbers Divisible by 2, 4, and 6

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We decompose quantum versions of numbers of the form $2mM$, $4m$, and $6n$.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}},$$

be the 2nd quantization of x , so that

$$\begin{aligned} [2]_q^\sim &= q + q^{-1}, \\ [3]_q^\sim &= q^2 + 1 + q^{-2}. \end{aligned}$$

1. A Decomposition Of General Even Numbers, Quantized

Theorem 1. For $m, M \in \mathbb{Z}_{\geq 2}$, we have:

$$[2]_q^\sim [m]_q^\sim [M]_q^\sim = [M - m]_q^\sim + [M + m]_q^\sim + 2 \sum_{i=1}^{m-1} \binom{n}{k} [M - m + 2i]_q^\sim. \quad (1)$$

Proof. We are going to use the general formula:

$$[a]_q^\sim + [a + 2]_q^\sim + \dots + [a + 2(m-1)]_q^\sim = [m]_q^\sim [a + m - 1]_q^\sim,$$

for $a = M - m$ and $a = M - m + 2$.

Thus,

$$\begin{aligned} [M - m]_q^\sim + [M - m + 2]_q^\sim + \dots + [M - m + 2(m-1)]_q^\sim &= [m]_q^\sim [M - m + m - 1]_q^\sim \\ &= [m]_q^\sim [M - m + m - 1]_q^\sim \end{aligned} \quad (2)$$

$$[M - m + 2]_q^\sim + \dots + [M - m + 2(m-1)]_q^\sim + [M + m]_q^\sim = [m]_q^\sim [M + 1]_q^\sim. \quad (3)$$

The RHS of (1) is the sum of (2) and (3). The useful formula

$$[a]_q^\sim + [a + 2]_q^\sim = [2]_q^\sim [a + 1]_q^\sim.$$

returns

$$[m]_q^\sim [M - 1]_q^\sim + [m]_q^\sim [M + 1]_q^\sim = [m]_q^\sim [2]_q^\sim [M]_q^\sim, \quad (4)$$

which is exactly the left-hand side of (1). ■

2. Decomposition Of Numbers Divisible by Four, Quantized

Theorem 2. For every $m \in \mathbb{Z}$, we have

$$([2]_q^\sim)^2 [m]_q^\sim = [m-2]_q^\sim + 2[m]_q^\sim + [m+2]_q^\sim. \quad (5)$$

Proof. We are going to use the easy formula:

$$[a]_q^\sim + [a+2]_q^\sim = [2]_q^\sim [a+1]_q^\sim. \quad (6)$$

Thus,

$$[m-2]_q^\sim + [m]_q^\sim = [2]_q^\sim [m-1]_q^\sim, \quad (7)$$

$$[m]_q^\sim + [m+2]_q^\sim = [2]_q^\sim [m+1]_q^\sim. \quad (8)$$

Adding up (7) and (8), the right-hand side of (5) becomes:

$$[2]_q^\sim [m-1]_q^\sim + [2]_q^\sim [m+1]_q^\sim = [2]_q^\sim [2]_q^\sim [m]_q^\sim,$$

which is the left-hand side of (5). ■

3. Decomposition of Numbers Divisible By 6, Quantized

Theorem 3. For every $m \in \mathbb{Z}$, we have

$$[2]_q^\sim [3]_q^\sim [m]_q^\sim = [m-3]_q^\sim + 2[m-1]_q^\sim + 2[m+1]_q^\sim + [m+3]_q^\sim. \quad (9)$$

Proof. We are going to use the two convenient formulae:

$$[a]_q^\sim + [a+2]_q^\sim = [2]_q^\sim [a+1]_q^\sim, \quad \forall a, \quad (10)$$

$$[a]_q^\sim + [a+2]_q^\sim + [a+4]_q^\sim = [3]_q^\sim [a+2]_q^\sim,$$

Thus, we have:

$$[m-3]_q^\sim + [m-1]_q^\sim + [m+1]_q^\sim = [3]_q^\sim [m-1]_q^\sim, \quad (11)$$

$$[m-1]_q^\sim + [m+1]_q^\sim + [m+3]_q^\sim = [3]_q^\sim [m+1]_q^\sim, \quad (12)$$

$$[m-1]_q^\sim + [m+1]_q^\sim = [2]_q^\sim [m]_q^\sim, \quad (13)$$

The sum of (11) through (13), which is the right-hand side of (9), gives us exactly the left-hand side of (9). ■